

# BLACK-SCHOLES IN A CEV RANDOM ENVIRONMENT: A NEW APPROACH TO SMILE MODELLING

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**ABSTRACT.** Classical (Itô diffusions) stochastic volatility models are not able to capture the steepness of small-maturity implied volatility smiles. Jumps, in particular exponential Lévy and affine models, which exhibit small-maturity exploding smiles, have historically been proposed to remedy this (see [58] for an overview). A recent breakthrough was made by Gatheral, Jaisson and Rosenbaum [30], who proposed to replace the Brownian driver of the instantaneous volatility by a short-memory fractional Brownian motion, which is able to capture the short-maturity steepness while preserving path continuity. We suggest here a different route, randomising the Black-Scholes variance by a CEV-generated distribution, which allows us to modulate the rate of explosion (through the CEV exponent) of the implied volatility for small maturities. The range of rates includes behaviours similar to exponential Lévy models and fractional stochastic volatility models. As a by-product, we make a conjecture on the small-maturity forward smile asymptotics of stochastic volatility models, in exact agreement with the results in [41] for the Heston model.

## 1. INTRODUCTION

We propose a simple model with continuous paths for stock prices that allows for small-maturity explosion of the implied volatility smile. It is indeed a well-documented fact on Equity markets (see for instance [29, Chapter 5]) that standard (Itô) stochastic models with continuous paths are not able to capture the observed steepness of the left wing of the smile when the maturity becomes small. To remedy this, several authors have suggested the addition of jumps, either in the form of an independent Lévy process or within the more general framework of affine diffusions. Jumps (in the stock price dynamics) imply an explosive behaviour for the small-maturity smile and are better able to capture the observed steepness of the small-maturity implied volatility smile. In particular, Tankov [58] showed that, for exponential Lévy models with Lévy measure supported on the whole real line, the squared implied volatility smile explodes as  $\sigma_\tau^2(k) \sim -k^2/(2\tau \log \tau)$ , as the maturity  $\tau$  tends to zero, where  $k$  represents the log-moneyness.

Gatheral, Jaisson and Rosenbaum [30] have recently been revisiting stochastic volatility models, where the instantaneous variance process is driven by a fractional Brownian motion. They suggest that the Hurst exponent should not be used as an indicator of the historical memory of the volatility, but rather as an additional parameter to be calibrated to the volatility surface. Their study reveals that  $H \in (0, 1/2)$  (in fact  $H \approx 0.1$  in their calibration results), indicating short memory of the volatility, thereby contradicting decades of time series analyses. By considering a specific fractional uncorrelated volatility model, directly inspired by the fractional version of the Heston model [15, 34], Guennoun, Jacquier and Roome [33] provide a theoretical

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justification of this result. They show in particular that, when  $H \in (0, 1/2)$ , the implied volatility explodes as  $\sigma_\tau^2(k) \sim y_0 \tau^{H-1/2} / \Gamma(H + 3/2)$  as  $\tau$  tends to zero (where  $y_0$  is the initial instantaneous variance).

In this paper we propose an alternative framework: we suppose that the stock price follows a standard Black-Scholes model; however the instantaneous variance, instead of being constant, is sampled from a continuous distribution. We first derive some general properties, interesting from a financial modelling point of view, and devote a particular attention to a particular case of it, where the variance is generated from independent CEV dynamics: Assume that interest rates and dividends are null, and let  $S$  denote the stock price process starting at  $S_0 = 1$ , the solution to the stochastic differential equation  $dS_\tau = S_\tau \sqrt{\mathcal{V}} dW_\tau$ , for  $\tau \geq 0$ , where  $W$  is a standard Brownian motion. Here,  $\mathcal{V}$  is a random variable, which we assume to be distributed as  $\mathcal{V} \sim Y_t$ , for some  $t > 0$ , where  $Y$  is the unique strong solution of the CEV dynamics  $dY_u = \xi Y_u^p dB_u$ ,  $Y_0 > 0$  where  $p \in \mathbb{R}$ ,  $\xi > 0$  and  $B$  is an independent Brownian motion (see Section 2.1 for precise statements). The main result of this paper (Theorem 2.3) is that the implied volatility generated from this model exhibits the following behaviour as the maturity  $\tau$  tends to zero:

$$(1.1) \quad \sigma_\tau^2(k) \sim \begin{cases} \frac{2(1-p)}{3-2p} \left( \frac{k^2 \xi^2 (1-p)t}{2\tau} \right)^{1/(3-2p)}, & \text{if } p < 1, \\ \frac{k^2 \xi^2 t}{\tau (\log \tau)^2}, & \text{if } p = 1, \\ \frac{k^2}{2(2p-1)\tau |\log \tau|}, & \text{if } p > 1, \end{cases}$$

for all  $k \neq 0$ . Sampling the initial variance from the CEV process at time  $t$  induces different time scales for small-maturity spot smiles, thereby providing flexibility to match steep small-maturity smiles. For  $p > 1$ , the explosion rate is the same as exponential Lévy models, and the case  $p \leq 1/2$  mimics the explosion rate of fractional stochastic volatility models. The CEV exponent  $p$  therefore allows the user to modulate the short-maturity steepness of the smile.

We are not claiming here that this model should come as a replacement of fractional stochastic volatility models or exponential Lévy models, notably because its dynamic structure looks too simple at first sight. However, we believe it can act as an efficient building block for more involved models, in particular for stochastic volatility models with initial random distribution for the instantaneous variance. While we leave these extensions for future research, we shall highlight how our model comes naturally into play when pricing forward-start options in stochastic volatility models. In [41] the authors proved that the small-maturity forward implied volatility smile explodes in the Heston model when the remaining maturity (after the forward-start date) becomes small. This explosion rate corresponds precisely to the case  $p = 1/2$  in (1.1). This in particular shows that the key quantity determining the explosion rate is the (right tail of the) variance distribution at the forward-start date (here corresponding to  $t$ ).

The paper is structured as follows: in Sections 2.1 and 2.2 we introduce our model and relate it to other existing approaches. In Section 2.3 we use the moment generating function to derive extreme strike asymptotics (for some special cases) and show why this approach is not readily applicable for small and large-maturity asymptotics. Sections 2.4 and 2.5 detail the main results, namely the small and large-maturity asymptotics of option prices and the corresponding implied volatility. Section 2.6 provides numerics and Section 2.7 describes the relationship between our model and the pricing of forward-start options in stochastic volatility models. Section 2.7 also includes a conjecture on the small-maturity forward smile in stochastic volatility models. Finally, the proofs of the main results are gathered in Section 3.

**Notations:** Throughout the paper, the  $\sim$  symbol means asymptotic equivalence, namely, the ratio of the left-hand side to the right-hand side tends to one.

## 2. MODEL AND MAIN RESULTS

**2.1. Model description.** We consider a filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_s)_{s \geq 0}, \mathbb{P})$  supporting a standard Brownian motion, and let  $(Z_s)_{s \geq 0}$  denote the solution to the following stochastic differential equation:

$$(2.1) \quad dZ_s = -\frac{1}{2}\mathcal{V}ds + \sqrt{\mathcal{V}}dW_s, \quad Z_0 = 0,$$

where  $\mathcal{V}$  is some random variable, independent of the Brownian motion  $W$ . The process  $(Z_s)_{s \geq 0}$ , in finance, corresponds to the logarithm of the underlying stock price. This is of course a simple example of a stochastic differential equation with random coefficients, existence and uniqueness of which were studied by Kohatsu-Higa, León and Nualart [43] and Alòs, León and Nualart [2]. In the case where  $\mathcal{V}$  is a discrete random variable, this model reduces to the mixture of distributions, analysed, in the Gaussian case by Brigo and Mercurio [10, 11]. In Appendix B, we discuss how to hedge in this model. In a stochastic volatility model where the instantaneous variance process  $(V_t)_{t \geq 0}$  is uncorrelated with the asset price process, the mixing result by Romano and Touzi [54] implies that the price of a European option with maturity  $\tau$  is the same as the one evaluated from the SDE (2.1) with  $\mathcal{V} = \tau^{-1} \int_0^\tau V_s ds$ . As  $\tau$  tends to zero, the distribution of  $\mathcal{V}$  approaches a Dirac Delta centred at the initial variance  $V_0$ . Asymptotics of the implied volatility are well known and weaknesses of classical stochastic volatility models are well documented [29]. Although such models fit into the framework of (2.1), we will not consider them further in this paper. Define pathwise the process  $M$  by  $M_s := -\frac{1}{2}s + W_s$  and let  $(\mathcal{T}_s)_{s \geq 0}$  be given by  $\mathcal{T}_s := s\mathcal{V}$ . Then  $\mathcal{T}$  is an independent increasing time-change process and  $Z = M_{\mathcal{T}}$ . In this way our model can be thought of as a random time change. Let now  $N$  be a Lévy process such that  $(e^{N_s})_{s \geq 0}$  is a  $(\mathcal{F}_s)_{s \geq 0}$ -adapted martingale; define  $\mathcal{V} := \tau^{-1} \int_0^\tau V_s ds$  where  $V$  is a positive and independent process, then  $(e^{N_{\mathcal{T}_s}})_{s \geq 0}$  is a classical time-changed exponential Lévy process, and pricing vanilla options is standard [16, Section 15.5]. However, as the maturity  $\tau$  tends to zero,  $\mathcal{V}$  converges in distribution to a Dirac Delta, in which case asymptotics are well known [58].

The model (2.1) is also related to the Uncertain Volatility Model of Avellaneda and Parás [3] (see also [19, 38, 48]), in which the Black-Scholes volatility is allowed to evolve randomly within two bounds. In this framework, sub-and super-hedging strategies (corresponding to best and worst case scenarios) are usually derived via the Black-Scholes-Barenblatt equation, and Fouque and Ren [27] recently provided approximation results when the two bounds become close to each other. One can also, at least formally, look at (2.1) from the perspective of fractional stochastic volatility models, first proposed by Comte et al. in [14], and later developed and revived in [15, 28, 30, 33]. In these models, standard stochastic volatility models are generalised by replacing the Brownian motion driving the instantaneous volatility by a fractional Brownian motion. This preserves the martingale property of the stock price process, and allows, in the case of short memory (Hurst parameter  $H$  between 0 and  $1/2$ ) for short-maturity steep skew of the implied volatility smile. However, the Mandelbrot-van Ness representation [49] of the fractional Brownian motion reads

$$W_t^H := \int_0^t \frac{dW_s}{(t-s)^\gamma} + \int_{-\infty}^0 \left( \frac{1}{(t-s)^\gamma} - \frac{1}{(-s)^\gamma} \right) dW_s,$$

for all  $t \geq 0$ , where  $\gamma := 1/2 - H$ . This representation in particular indicates that, at time zero, the instantaneous variance, being driven by a fractional Brownian motion, incorporates some randomness (through the second

integral). Finally, we agree that, at first sight, randomising the variance may sound unconventional. As mentioned in the introduction, we see this model as a building block for more involved models, in particular stochastic volatility with random initial variance, the full study of which is the purpose of ongoing research. After all, market data only provides us with an initial value of the stock price, and the initial level of the variance is unknown, usually let as a parameter to calibrate. In this sense, it becomes fairly natural to leave the latter random.

The framework constituted by the stochastic differential equation (2.1) is a simple case of a diffusion in random environment. We refer the interested reader to the seminal paper by Papanicolaou and Varadhan [53], the monographs by Komorowski et al. [44], by Sznitman [57], and the lectures notes by Bolthausen and Sznitman [8] and by Zeitouni [60]. The classical set-up (say in  $\mathbb{R}^d$ ) is that of a given probability space  $(\tilde{\Omega}, \mathcal{A}, \mathbb{Q})$  describing the random environment and a group of transformations  $(\tau_x)_{x \in \mathbb{R}^d}$ , jointly measurable in  $x \in \mathbb{R}^d$  and  $\omega \in \tilde{\Omega}$  (the transformation essentially indicates a translation of the environment in the  $x$ -direction). Consider now two functions  $b, a : \tilde{\Omega} \rightarrow \mathbb{R}^d$  and define

$$b(x, \omega) := b(\tau_x(\omega)) \quad \text{and} \quad a(x, \omega) := a(\tau_x(\omega)), \quad \text{for all } x \in \mathbb{R}^d, \omega \in \tilde{\Omega}.$$

For each  $x \in \mathbb{R}^d$  and  $\omega \in \tilde{\Omega}$ , we let  $\mathbb{Q}_{x, \omega}$  denote the unique solution to the martingale problem starting at  $x$  and associated to the differential operator  $\mathcal{L}^\omega := \frac{1}{2}a(\cdot, \omega)\Delta + b(\cdot, \omega)\nabla$ . The probability law  $\mathbb{Q}_{x, \omega}$  is called the *quenched law*, and one can define the solution to the corresponding stochastic differential equation  $\mathbb{Q}_{x, \omega}$ -almost surely. The *annealed law* is the semi-product  $\mathbb{Q}_x := \mathbb{Q} \times \mathbb{Q}_{x, \omega}$ , and corresponds to averaging over the random environment. Most of the results in the literature, using the method of the environment viewed from the particle, however, impose Lipschitz continuity on the drift  $b(\cdot)$  and the diffusion coefficient  $a(\cdot)$ , and uniform ellipticity of  $a(\cdot)$ . These conditions clearly do not hold for (2.1), where  $b(\cdot, \omega) \equiv -\frac{1}{2}\omega$  and  $a(\cdot, \omega) \equiv \omega$ , since  $\omega$  takes values in  $[0, \infty)$ . We shall leave more precise details and applications of random environment to future research. As far as we are aware, this framework has not been applied yet in the option pricing literature, the closest being the recent publication by Spiliopoulos [56], who proves quenched (almost sure with respect to the environment) large deviations for a multi-scale diffusion (in a certain regime), assuming stationarity and ergodicity of the random environment. Interestingly, diffusions (specifically Ornstein-Uhlenbeck processes) in random environments have been justified by Föllmer and Schweizer [23] and by Horst [35] (see also [52]) as the continuous-time limit of discrete-time dynamics for the asset price defined as the outcome of equilibria determined by the interaction of informed and noise traders.

**2.1.1. Moment generating function.** In [24, 25, 37], the authors used the theory of large deviations, and in particular the Gärtner-Ellis theorem, to prove small-and large-maturity behaviours of the implied volatility in the Heston model and more generally (in [37]) for affine stochastic volatility models. This approach relies solely on the knowledge of the cumulant generating function of the underlying stock price, and its rescaled limiting behaviour. For any  $\tau \geq 0$ , let  $\Lambda^Z(u, \tau) := \log \mathbb{E}(e^{uZ_\tau})$  denote the cumulant generating function of  $Z_\tau$ , defined on the effective domain  $\mathcal{D}_\tau^Z := \{u \in \mathbb{R} : |\Lambda^Z(u, \tau)| < \infty\}$ ; similarly denote  $\Lambda^\mathcal{V}(u) \equiv \log \mathbb{E}(e^{u\mathcal{V}})$ , whenever it is well defined. A direct application of the tower property for expectations yields

$$(2.2) \quad \Lambda^Z(u, \tau) = \Lambda^\mathcal{V}\left(\frac{u(u-1)\tau}{2}\right), \quad \text{for all } u \in \mathcal{D}_\tau^Z.$$

Unfortunately, the cumulant generating function of  $\mathcal{V}$  is not available in closed-form in general. In Section 2.3 below, we shall see some examples where such a closed-form solution is available, and where direct computations

are therefore possible. We note in passing that this simple representation allows, at least in principle, for straightforward (numerical) computations of the slopes of the wings of the implied volatility using Roger Lee's Moment Formula [45] (see also Section 2.3.2). The latter are indeed given directly by the boundaries (in  $\mathbb{R}$ ) of the effective domain of  $\Lambda^\nu$ . Note further that the model (2.1) could be seen as a time-changed Brownian motion (with drift); the representation (2.2) clearly rules out the case where  $Z$  is a simple exponential Lévy process (in which case  $\Lambda^Z(u, \tau)$  would be linear in  $\tau$ ). In view of Roger Lee's formula, this also implies that, contrary to the Lévy case, the slopes of the implied volatility wings are not constant over time in our model.

**2.2. CEV randomisation.** As mentioned above, this paper is a first step towards the introduction of 'random environment' into the realm of option pricing, and we believe that, seeing it 'at work' through a specific, yet non-trivial, example, will speak for its potential prowess. We assume from now on that  $\mathcal{V}$  corresponds to the distribution of the random variable generated, at some time  $t$ , by the solution to the CEV stochastic differential equation  $dY_u = \xi Y_u^p dB_u$ ,  $Y_0 = y_0 > 0$  where  $p \in \mathbb{R}$ ,  $\xi > 0$  and  $B$  is a standard Brownian motion, independent of  $W$ . The CEV process [9, 42] is the unique strong solution to this stochastic differential equation, up to the stopping time  $\tau_0^Y := \inf_{u>0} \{Y_u = 0\}$ . The behaviour of the process after  $\tau_0^Y$  depends on the value of  $p$ , and shall be discussed below. We let  $\Gamma(n; x) := \Gamma(n)^{-1} \int_0^x t^{n-1} e^{-t} dt$  denote the normalised lower incomplete Gamma function, and  $m_t := \mathbb{P}(Y_t = 0) = \mathbb{P}(\mathcal{V} = 0)$  represent the mass at the origin. Straightforward computations show that, whenever the origin is an absorbing boundary, the density  $\zeta_p(y) \equiv \mathbb{P}(Y_t \in dy)/dy$  is norm decreasing and ( $\eta$  defined in (2.4))

$$(2.3) \quad m_t = 1 - \Gamma\left(-\eta; \frac{y_0^{2(1-p)}}{2\xi^2(1-p)^2 t}\right) > 0;$$

otherwise  $m_t = 0$  and the density  $\zeta_p$  is norm preserving. When  $p \in [1/2, 1)$ , the origin is naturally absorbing. When  $p \geq 1$ , the process  $Y$  never hits zero  $\mathbb{P}$ -almost surely. Finally, when  $p < 1/2$ , the origin is an attainable boundary, and can be chosen to be either absorbing or reflecting. Absorption is compulsory if  $Y$  is required to be a martingale [36, Chapter III, Lemma 3.6]. Here it is only used as a building block for the instantaneous variance, and such a requirement is therefore not needed, so that both cases (absorption and reflection) will be treated. Define the constants

$$(2.4) \quad \eta := \frac{1}{2(p-1)}, \quad \mu := \log(y_0) - \frac{\xi^2 t}{2},$$

and the function  $\varphi_\eta : (0, \infty) \rightarrow (0, \infty)$  by

$$\varphi_\eta(y) := \frac{y_0^{1/2} y^{1/2-2p}}{|1-p|\xi^2 t} \exp\left(-\frac{y^{2(1-p)} + y_0^{2(1-p)}}{2\xi^2 t(1-p)^2}\right) I_\eta\left(\frac{(y_0 y)^{1-p}}{(1-p)^2 \xi^2 t}\right),$$

where  $I_\eta$  is the modified Bessel function of the first kind of order  $\eta$  [1, Section 9.6]. The CEV density,  $\zeta_p(y) := \mathbb{P}(Y_t \in dy)/dy$ , then reads

$$(2.5) \quad \zeta_p(y) = \begin{cases} \varphi_{-\eta}(y), & \text{if } p \in [1/2, 1) \text{ or } p < \frac{1}{2} \text{ with absorption,} \\ \varphi_\eta(y), & \text{if } p > 1 \text{ or } p < \frac{1}{2} \text{ with reflection,} \\ \frac{1}{y\xi\sqrt{2\pi t}} \exp\left(-\frac{(\log(y) - \mu)^2}{2\xi^2 t}\right), & \text{if } p = 1, \end{cases}$$

valid for  $y \in (0, \infty)$ . When  $p \geq 1$ , the density  $\zeta_p$  converges to zero around the origin, implying that paths are being pushed away from the origin. On the other hand  $\zeta_p$  diverges to infinity at the origin when  $p < 1/2$ , so that the paths have a propensity towards the vicinity of the origin.

**2.3. The moment generating function approach.** In the literature on implied volatility asymptotics, the moment generating function of the stock price has proved to be an extremely useful tool to obtain sharp estimates. This is obviously the case for the wings of the smile (small and large strikes) via Roger Lee's formula, mentioned in Section 2.1.1, but also to describe short- and large-maturity asymptotics, as developed for instance in [37] or [39], via the use of (a refined version of) the Gärtner-Ellis theorem. As shown in Section 2.1.1, the moment generating function of a stock price satisfying (2.1) is fully determined by that of the random variable  $\mathcal{V}$ . However, even though the density of the latter is known in closed form (Equation (2.5)), the moment generating function is not so for general values of  $p$ . In the cases  $p = 0$  (with either reflecting or absorbing boundary) and  $p = 1/2$ , a closed-form expression is available and direct computations are possible.

**2.3.1. Computation of the moment generating function.** Denote by  $\Lambda_{0,r}^{\mathcal{V}}$ ,  $\Lambda_{0,a}^{\mathcal{V}}$  and  $\Lambda_{1/2}^{\mathcal{V}}$  the moment generating function of the random variable  $\mathcal{V}$  when  $p = 0$  (the subscript 'r' / 'a' denotes the reflecting / absorbing behaviour at the origin) and  $p = 1/2$ . Straightforward computations yield

$$(2.6) \quad \begin{aligned} \Lambda_{0,a}^{\mathcal{V}}(u) &= \log \left[ m_t + \frac{1}{2} \exp \left( \frac{(u\xi^2 t - 2y_0)u}{2} \right) \left\{ e^{2uy_0} \mathcal{E} \left( \frac{u\xi^2 t + y_0}{\xi\sqrt{2t}} \right) + e^{2uy_0} - 1 - \mathcal{E} \left( \frac{u\xi^2 t - y_0}{\xi\sqrt{2t}} \right) \right\} \right], \\ \Lambda_{0,r}^{\mathcal{V}}(u) &= \log \left[ \frac{1}{2} \exp \left( \frac{(u\xi^2 t - 2y_0)u}{2} \right) \left\{ e^{2uy_0} \mathcal{E} \left( \frac{u\xi^2 t + y_0}{\xi\sqrt{2t}} \right) + e^{2uy_0} + 1 + \mathcal{E} \left( \frac{u\xi^2 t - y_0}{\xi\sqrt{2t}} \right) \right\} \right], \\ \Lambda_{1/2}^{\mathcal{V}}(u) &= \frac{2y_0 u}{2 - u\xi^2 t}, \end{aligned}$$

where  $\mathcal{E}(z) \equiv \frac{2}{\sqrt{\pi}} \int_0^z \exp(-x^2) dx$  is the error function. Note that when  $p = 1/2$  and  $p = 0$  in the absorption case, one needs to take into account the mass at zero in (2.3) when computing these expectations.

**2.3.2. Roger Lee's wing formula.** In [45], Roger Lee provided a precise link between the slope of the total implied variance in the wings and the boundaries of the domain of the moment generating function of the stock price. More precisely, for any  $\tau \geq 0$ , let  $u_+(\tau)$  and  $u_-(\tau)$  be defined as

$$u_+(\tau) := \sup\{u \geq 1 : |\Lambda^Z(u, \tau)| < \infty\} \quad \text{and} \quad u_-(\tau) := \sup\{u \geq 0 : |\Lambda^Z(-u, \tau)| < \infty\}.$$

The implied volatility  $\sigma_\tau(k)$  then satisfies

$$\limsup_{k \uparrow \infty} \frac{\sigma_\tau(k)^2 \tau}{k} = \psi(u_+(\tau) - 1) =: \beta_+(\tau) \quad \text{and} \quad \limsup_{k \downarrow -\infty} \frac{\sigma_\tau(k)^2 \tau}{|k|} = \psi(u_-(\tau)) =: \beta_-(\tau),$$

where the function  $\psi$  is defined by  $\psi(u) = 2 - 4 \left( \sqrt{u(u+1)} - u \right)$ . Combining (2.6) and (2.2) yields a closed-form expression for the moment generating function of the stock price when  $p \in \{0, 1/2\}$ . It is clear that, when  $p = 0$ ,  $u_\pm(\tau) = \pm\infty$  for any  $\tau \geq 0$ , and hence the slopes of the left and right wings are equal to zero (the total variance flattens for small and large strikes). In the case where  $p = 1/2$ , explosion will occur as soon as  $(\frac{1}{2}u(u-1)\tau\xi^2 t - 2) = 0$ , so that

$$u_\pm(\tau) = \frac{1}{2} \pm \frac{1}{2} \sqrt{1 + \frac{16}{\xi^2 t \tau}}, \quad \text{and} \quad \beta_-(\tau) = \beta_+(\tau) = \frac{2}{\xi\sqrt{t\tau}} \left( \sqrt{\xi^2 t \tau + 16} - 4 \right), \quad \text{for all } \tau > 0.$$

The left and right slopes are the same, but the product  $\xi^2 t$  can be directly calibrated on the observed wings. Note that the map  $\tau \mapsto \beta_\pm(\tau)$  is concave and increasing from 0 to 2. In [20, 21], the authors highlighted some symmetry properties between the small-time behaviour of the smile and its tail asymptotics. We obtain here some interesting asymmetry, in the sense that one can observe the same type of rate of explosion (power behaviour, given by (1.1) in the case  $p < 1$ ), but different tail behaviour for fixed maturity. As  $\tau$  tends to infinity,  $\beta_\pm(\tau)$  converges to 2, so that the implied volatility smile does not 'flatten out', as is usually the case



for Itô diffusions or affine stochastic volatility models (see for instance [37]). In Section 2.5 below, we make this more precise by investigating the large-time behaviour of the implied volatility using the density of the CEV-distributed variance.

**2.3.3. Small-time asymptotics.** In order to study the small-maturity behaviour of the implied volatility, one could, whenever the moment generating function of the stock price is available in closed form (e.g. in the case  $p \in \{0, 1/2\}$ ), apply the methodology developed in [24]. The latter is based on the Gärtner-Ellis theorem, which, essentially, consists of finding a smooth convex pointwise limit (as  $\tau$  tends to zero) of some rescaled version of the cumulant generating function. In the case where  $p = 1/2$ , it is easy to show that

$$(2.7) \quad \lim_{\tau \downarrow 0} \tau^{1/2} \log \Lambda^Z \left( \frac{u}{\sqrt{\tau}}, \tau \right) = \begin{cases} 0, & \text{if } u \in \left( -\frac{2}{\xi\sqrt{t}}, \frac{2}{\xi\sqrt{t}} \right), \\ +\infty, & \text{otherwise.} \end{cases}$$

The nature of this limiting behaviour falls outside the scope of the Gärtner-Ellis theorem, which requires strict convexity of the limiting rescaled cumulant generating function. It is easy to see that any other rescaling would yield even more degenerate behaviour. One could adapt the proof of the Gärtner-Ellis theorem, as was done in [41] for the small-maturity behaviour of the forward implied volatility smile in the Heston model (see also [17] and references therein for more examples of this kind). In the case (2.7), we are exactly as in the framework of [41], in which the small-maturity smile (squared) indeed explodes as  $\tau^{-1/2}$ , precisely the same explosion as the one in (1.1). Unfortunately, as we mentioned above, the moment generating function of the stock price is not available in general, and this approach is hence not amenable here.

**2.3.4. Large-time asymptotics.** The analysis above, based on the moment generating function of the stock price, can be carried over to study the large-time behaviour of the latter. In the case  $p = 1/2$ , computations are fully explicit, and the following pointwise limit follows from simple straightforward manipulations:

$$\lim_{\tau \uparrow \infty} \tau^{-1} \Lambda^Z(u, \tau) = \begin{cases} 0, & \text{if } u \in [0, 1], \\ +\infty, & \text{otherwise.} \end{cases}$$

The nature of this asymptotic behaviour, again, falls outside the scope of standard large deviations analysis, and tedious work, in the spirit of [5, 41], would be needed to pursue this route.

**2.4. Small-time behaviour of option prices and implied volatility.** In the Black-Scholes model  $dS_t = \sqrt{w} S_t dW_t$  starting at  $S_0 = 1$ , a European call option with strike  $e^k$  and maturity  $T > 0$  is worth

$$(2.8) \quad \text{BS}(k, w, T) = \mathcal{N} \left( -\frac{k}{\sqrt{wT}} + \frac{\sqrt{wT}}{2} \right) - e^k \mathcal{N} \left( -\frac{k}{\sqrt{wT}} - \frac{\sqrt{wT}}{2} \right).$$

For any  $k \in \mathbb{R} \setminus \{0\}$ ,  $T > 0$ , and  $p > 1$ , the quantity

$$(2.9) \quad J^p(k) := \begin{cases} \int_0^\infty \text{BS} \left( k, \frac{y}{T}, T \right) y^{-p} dy, & \text{if } k > 0, \\ \int_0^\infty \left( e^k - 1 + \text{BS} \left( k, \frac{y}{T}, T \right) \right) y^{-p} dy, & \text{if } k < 0, \end{cases}$$

is clearly independent of  $T$  and is well defined. Indeed, since the stock price is a martingale starting at one, Call options are always bounded above by one, and hence, for  $k > 0$ ,  $J^p(k) \leq \int_0^1 \text{BS}(k, y/T, T) y^{-p} dy + \int_1^\infty y^{-p} dy$ . The second integral is finite since  $p > 1$ . When  $k > 0$ , the asymptotic behaviour

$$\text{BS} \left( k, \frac{y}{T}, T \right) \sim \exp \left( -\frac{k^2}{2y} + \frac{k}{2} \right) \frac{y^{3/2}}{k^2 \sqrt{2\pi}}$$

holds as  $y$  tends to zero, so that  $\lim_{y \downarrow 0} \text{BS}(k, y/T, T)y^{-p} = 0$ , and hence the integral is finite. A similar analysis holds when  $k < 0$  and using put-call parity. Define now the following constants:

$$(2.10) \quad \beta_p := \frac{1}{3-2p}, \quad \bar{y}_p := \left( \frac{k^2 \xi^2 t (1-p)}{2} \right)^{\beta_p}, \quad y^* := \frac{k^2 \xi^2 t}{2},$$

the first two only when  $p < 1$ , and note that  $\beta_p \in (0, 1)$ ; define further the following functions from  $(0, \infty)$  to  $\mathbb{R}$ :

$$(2.11) \quad \begin{cases} f_0(y) &:= \frac{k^2}{2y} + \frac{y^{2(1-p)}}{2\xi^2 t (1-p)^2}, & f_1(y) &:= \frac{(y y_0)^{(1-p)}}{\xi^2 t (1-p)^2}, \\ g_0(y) &:= \frac{k^2}{2y} + \frac{\log(y)}{\xi^2 t}, & g_1(y) &:= \frac{\log(y)}{\xi^2 t}, \end{cases}$$

as well as the following ones, parameterised by  $p$ :

	$p < 1$	$p = 1$	$p > 1$
$c_1(t, p)$	$f_0(\bar{y}_p)$	$1/(2\xi^2 t)$	0
$c_2(t, p)$	$f_1(\bar{y}_p)$	$1/(2\xi^2 t)$	0
$c_3(t, p)$	$\frac{6-5p}{6-4p}$	$g_0(y^*) - \frac{\mu}{\xi^2 t}$	$2p-1$
$c_4(t, p)$	0	$g_1(y^*) - \frac{\mu}{\xi^2 t} - 2$	0
$c_5(t, p)$	$\frac{y_0^{\frac{p}{2}} \bar{y}_p^{\frac{3}{2}(1-p)} \exp\left(\frac{k}{2} - \frac{y_0^{2(1-p)}}{2\xi^2 t (1-p)^2} + \frac{f_1(\bar{y}_p)^2}{2f_0'(\bar{y}_p)}\right)}{k^2 \xi \sqrt{2\pi f_0''(\bar{y}_p)} t}$	$\frac{\exp\left(\frac{k}{2} - \frac{\mu^2}{2\xi^2 t} + \frac{\mu \log(y^*)}{\xi^2 t}\right)}{4\sqrt{\pi} k ^{-1}\xi^{-3}t^{-3/2}}$	$\frac{2(p-1)e^{-\frac{y_0^{2(p-1)}}{2\xi^2 t (1-p)^2}} J^{2p}(k)}{(2(1-p)^2 \xi^2 t)^{\eta+1} \Gamma(\eta+1)}$
$h_1(\tau, p)$	$\tau^{\beta_p-1}$	$(\log(\tau) + \log \log(\tau^{-1}))^2$	0
$h_2(\tau, p)$	$\tau^{(\beta_p-1)/2}$	$\frac{(\log  \log(\tau) )^2}{ \log(\tau) }$	0
$\mathcal{R}(\tau, p)$	$\mathcal{O}\left(\tau^{(1-\beta_p)/2}\right)$	$\mathcal{O}\left(\frac{1}{ \log(\tau) }\right)$	$\mathcal{O}(\tau^{p-1})$

TABLE 1. List of constants and functions

The following theorem (proved in Section 3.1) is the central result of this paper (although its equivalent below, in terms of implied volatility, is more informative for practical purposes):

**Theorem 2.1.** *The following expansion holds for all  $k \in \mathbb{R} \setminus \{0\}$  as  $\tau$  tends to zero:*

$$\mathbb{E}(e^{Z_\tau} - e^k)^+ = (1 - e^k)^+ + \exp\left(-c_1(t, p)h_1(\tau, p) + c_2(t, p)h_2(\tau, p)\right) \tau^{c_3(t, p)} |\log(\tau)|^{c_4(t, p)} c_5(t, p) [1 + \mathcal{R}(\tau, p)].$$

**Remark 2.2.**

- (i) Whenever  $p \leq 1$ ,  $c_1$  and  $c_2$  are strictly positive; the function  $c_5$  is always strictly positive; when  $p < 1$ ,  $c_3$  is strictly positive; when  $p = 1$ , the functions  $c_3$  and  $c_4$  can take positive and negative values;
- (ii) Whenever  $p \leq 1$ ,  $h_2(\tau, p) \leq h_1(\tau, p)$  for  $\tau$  small enough, so that the leading order is provided by  $h_1$ ;
- (iii) In the lognormal case  $p = 1$ ,  $h_1(\tau, 1) \sim (\log \tau)^2$  as  $\tau$  tends to zero, so that the exponential decay of option prices is governed at leading order by  $\exp(-c_1(t, 1)(\log \tau)^2)$ .

Using Theorem 2.1 and small-maturity asymptotics for the Black-Scholes model (see [26, Corollary 3.5] or [31]), it is straightforward to translate option price asymptotics into asymptotics of the implied volatility:



**Theorem 2.3.** *For any  $k \in \mathbb{R} \setminus \{0\}$ , the small-maturity implied volatility smile behaves as follows:*

$$\sigma_\tau^2(k) \sim \begin{cases} (1 - \beta_p) \left( \frac{k^2 \xi^2 t (1 - p)}{2\tau} \right)^{\beta_p}, & \text{if } p < 1, \\ \frac{k^2 \xi^2 t}{\tau \log(\tau)^2}, & \text{if } p = 1, \\ \frac{k^2}{2(2p - 1)\tau |\log(\tau)|}, & \text{if } p > 1. \end{cases}$$

This theorem only presents the leading-order asymptotic behaviour of the implied volatility as the maturity becomes small. One could in principle (following [13] or [31]) derive higher-order terms, but these additional computations would impact the clarity of this singular behaviour. In the at-the-money  $k = 0$  case, the implied volatility converges to a constant:

**Lemma 2.4.** *The at-the-money implied volatility  $\sigma_\tau(0)$  converges to  $\mathbb{E}(\sqrt{V})$  as  $\tau$  tends to zero.*

The proof of the lemma follows steps analogous to [41, Lemma 4.3], and we omit the details here. It in fact does not depend on any particular form of distribution of  $\sqrt{V}$ , as long as the expectation exists. Note that, from Theorem 2.3, as  $p$  approaches 1 from below, the rate of explosion approaches  $\tau^{-1}$ . When  $p$  tends to 1 from above, the explosion rate is  $1/(\tau |\log \tau|)$  instead. So there is a "discontinuity" at  $p = 1$  and the actual rate of explosion is less than both these limits. As an immediate consequence of Theorem 2.1 we have the following corollary. Define the following functions:

$$h^*(\tau, p) := \begin{cases} \tau^{1-\beta_p}, & \text{if } p < 1, \\ |\log(\tau)^{-1}|, & \text{if } p > 1, \\ \log(\tau)^{-2}, & \text{if } p = 1, \end{cases} \quad \text{and} \quad \Lambda_p^*(k) := \begin{cases} c_1(t, p), & \text{if } p \leq 1, \\ 2p - 1, & \text{if } p > 1, \end{cases}$$

where  $c_1(t, p)$  is defined in Table 1, and depends on  $k$  (through  $\bar{y}_p$ ).

**Corollary 2.5.** *For any  $p \in \mathbb{R}$ , the sequence  $(Z_\tau)_{\tau \geq 0}$  satisfies a large deviations principle with speed  $h^*(\tau, p)$  and rate function  $\Lambda_p^*$  as  $\tau$  tends to zero. Furthermore, the rate function is good only when  $p < 1$ .*

Recall that a real-valued sequence  $(\mathfrak{Z}_n)_{n \geq 0}$  satisfies a large deviations principle (see [18] for a precise introduction to the topic) with speed  $n$  and rate function  $\Lambda^*$  if, for any Borel subset  $B \subset \mathbb{R}$ , the inequalities

$$-\inf_{z \in B^\circ} \Lambda^*(z) \leq \liminf_{n \uparrow \infty} n^{-1} \log \mathbb{P}(\mathfrak{Z}_n \in B) \leq \limsup_{n \uparrow \infty} n^{-1} \log \mathbb{P}(\mathfrak{Z}_n \in B) \leq -\inf_{z \in \bar{B}} \Lambda^*(z)$$

hold, where  $\bar{B}$  and  $B^\circ$  denote the closure and interior of  $B$  in  $\mathbb{R}$ . The rate function  $\Lambda^* : \mathbb{R} \rightarrow \mathbb{R} \cup \{+\infty\}$ , by definition, is a lower semi-continuous, non-negative and not identically infinite, function such that the level sets  $\{x \in \mathbb{R} : \Lambda^*(x) \leq \alpha\}$  are closed for all  $\alpha \geq 0$ . It is said to be a good rate function when these level sets are compact (in  $\mathbb{R}$ ).

*Proof.* The proof of Theorem 2.1 holds with only minor modifications for digital options, which are equivalent to probabilities of the form  $\mathbb{P}(Z_\tau \leq k)$  or  $\mathbb{P}(Z_\tau \geq k)$ . For  $p \in (-\infty, 1]$ , one can then show that

$$\lim_{\tau \downarrow 0} h^*(\tau, p) \log \mathbb{P}(Z_\tau \leq k) = -\inf \{ \Lambda_p^*(x) : x \leq k \}.$$

The infimum is null whenever  $k > 0$  and  $p < 1$ , and  $\Lambda_1^*(x) \equiv 1/(2\xi^2 t)$  is constant. Consider now an open interval  $(a, b) \subset \mathbb{R}$ . Since  $(a, b) = (-\infty, b) \setminus (-\infty, a]$ , then by continuity and convexity of  $\Lambda_p^*$ , we obtain

$$\lim_{\tau \downarrow 0} h^*(\tau, p) \log \mathbb{P}(Z_\tau \in (a, b)) = -\inf_{x \in (a, b)} \Lambda_p^*(x).$$

Since any Borel set of the real line can be written as a (countable) union / intersection of open intervals, the corollary follows from the definition of the large deviations principle [18, Section 1.2]. When  $p \in (1, \infty)$ , the only non-trivial choice of speed is  $|(\log \tau)^{-1}|$ , in which case  $\lim_{\tau \downarrow 0} |(\log \tau)^{-1}| \log \mathbb{P}(Z_\tau \leq k) = -(2p-1)$ . Clearly, the constant function is a rate function (the level sets, either the empty set or the real line, being closed in  $\mathbb{R}$ ), and the corollary follows.  $\square$

**Remark 2.6.** In the case  $p = 1/2$ , as discussed in Section 2.3.4, the moment generating function of  $Z$  is available in closed form. However, the large deviations principle does not follow from the Gärtner-Ellis theorem, since the pointwise rescaled limit of the mgf is degenerate (in the sense of (2.7)).

**2.4.1. Small-maturity at-the-money skew and convexity.** The goal of this section is to compute asymptotics for the at-the-money skew and convexity of the smile as the maturity becomes small. These quantities are useful to traders who actually observe them (or approximations thereof) on real data. We define the left and right derivatives by  $\partial_k^- \sigma_\tau^2(0) := \lim_{k \uparrow 0} \partial_k \sigma_\tau^2(k)|_{k=0}$  and  $\partial_k^+ \sigma_\tau^2(0) := \lim_{k \downarrow 0} \partial_k \sigma_\tau^2(k)|_{k=0}$ , and similarly  $\partial_{kk}^- \sigma_\tau^2(0) := \lim_{k \uparrow 0} \partial_{kk} \sigma_\tau^2(k)|_{k=0}$  and  $\partial_{kk}^+ \sigma_\tau^2(0) := \lim_{k \downarrow 0} \partial_{kk} \sigma_\tau^2(k)|_{k=0}$ . The following lemma describes this short-maturity behaviour in the general case where  $\mathcal{V}$  is any random variable supported on  $[0, \infty)$ .

**Lemma 2.7.** Consider (2.1) and assume that  $\mathbb{E}(\mathcal{V}^{n/2}) < \infty$  for  $n = -1, 1, 3$ , and  $m_t := \mathbb{P}(\mathcal{V} = 0) < 1$ . As  $\tau$  tends to zero,

$$\begin{aligned} \partial_k^- \sigma_\tau^2(0) &\sim -\frac{\mathbb{E}(\sqrt{\mathcal{V}})}{48} \left( \mathbb{E}(\mathcal{V}^{3/2}) - \mathbb{E}(\sqrt{\mathcal{V}})^3 \right) \tau - m_t \frac{\mathbb{E}(\sqrt{\mathcal{V}})\sqrt{\pi}}{\sqrt{2\tau}}, \\ \partial_k^+ \sigma_\tau^2(0) &\sim -\frac{\mathbb{E}(\sqrt{\mathcal{V}})}{48} \left( \mathbb{E}(\mathcal{V}^{3/2}) - \mathbb{E}(\sqrt{\mathcal{V}})^3 \right) \tau + m_t \frac{\mathbb{E}(\sqrt{\mathcal{V}})\sqrt{\pi}}{\sqrt{2\tau}}, \\ \partial_{kk}^- \sigma_\tau^2(0) &\sim \partial_{kk}^+ \sigma_\tau^2(0) \sim \frac{\mathbb{E}(\sqrt{\mathcal{V}})}{\tau} \left( \mathbb{E}(\mathcal{V}^{-1/2}) - \mathbb{E}(\sqrt{\mathcal{V}})^{-1} \left( 1 - \frac{m_t^2 \sqrt{\pi}}{8} \right) \right). \end{aligned}$$

Jensen's inequality and the fact that the support of  $\mathcal{V}$  is in  $[0, \infty)$  imply that both  $\mathbb{E}(\mathcal{V}^{3/2}) - \mathbb{E}(\sqrt{\mathcal{V}})^3$  and  $\mathbb{E}(\mathcal{V}^{-1/2}) - \mathbb{E}(\sqrt{\mathcal{V}})^{-1}$  are strictly positive. The small-maturity at-the-money skew is always negative for small  $m_t$ . This in particular means that the smile generated by (2.1) is not necessarily symmetric. When  $m_t > 0$ , the at-the-money left skew explodes to  $-\infty$  and the at-the-money right skew explodes to  $+\infty$ . Furthermore, the small-maturity at-the-money convexity tends to infinity.

*Proof.* We first focus on the at-the-money skew. By definition  $C(k, \tau) = \text{BS}(k, \sigma_\tau^2(k), \tau)$  and therefore

$$\partial_k C(k, \tau) = \partial_k \text{BS}(k, \sigma_\tau^2(k), \tau) + \partial_k \sigma_\tau^2(k) \partial_w \text{BS}(k, \sigma_\tau^2(k), \tau).$$

Also by (3.1), an immediate application of Fubini yields

$$(2.12) \quad \partial_k C(k, \tau) = \int_0^\infty \partial_k \text{BS}(k, y, \tau) \zeta_p(y) dy + m_t \partial_k (1 - e^k)^+,$$

We first assume that  $m_t = 0$ . The at-the-money skew is then given by

$$\partial_k \sigma_\tau^2(k)|_{k=0} = (\partial_w \text{BS}(k, \sigma_\tau^2(k), \tau)|_{k=0})^{-1} \left( \int_0^\infty \partial_k \text{BS}(k, y, \tau)|_{k=0} \zeta_p(y) dy - \partial_k \text{BS}(k, \sigma_\tau^2(k), \tau)|_{k=0} \right).$$

Recall now from Lemma 2.4 that  $\sigma_\tau(0) = \mathbb{E}(\sqrt{\mathcal{V}}) + o(1)$ . Straightforward computations then yield

$$(2.13) \quad \begin{cases} \partial_k \text{BS}(k, \sigma_\tau^2(k), \tau)|_{k=0} = -\mathcal{N}\left(\frac{\sigma_\tau(0)\sqrt{\tau}}{2}\right) = -\frac{1}{2} + \frac{\sigma_\tau(0)\sqrt{\tau}}{2\sqrt{2\pi}} - \frac{\sigma_\tau^3(0)\tau^{3/2}}{48\sqrt{2\pi}} + \mathcal{O}\left(\sigma_\tau^5(0)\tau^{5/2}\right), \\ \partial_w \text{BS}(k, \sigma_\tau^2(k), \tau)|_{k=0} = \frac{\sqrt{\tau}}{\sigma_\tau(0)\sqrt{2\pi}} \exp\left(-\frac{\sigma_\tau^2(0)\tau}{8}\right). \end{cases}$$

as  $\tau$  tends to zero. Hence

$$\partial_k \sigma_\tau^2(k)|_{k=0} = \exp\left(\frac{\sigma_\tau^2(0)\tau}{8}\right) \frac{\sigma_\tau(0)}{2} \left( \mathbb{E}(\sqrt{\mathcal{V}}) - \sigma_\tau(0) - \frac{(\mathbb{E}(\mathcal{V}^{3/2}) - \sigma_\tau(0)^3)\tau}{24} + \mathcal{O}(\sigma_\tau(0)^3\tau^5) \right),$$

and so, as  $\tau$  tends to zero,

$$(2.14) \quad \partial_k \sigma_\tau^2(k)|_{k=0} \sim -\frac{\mathbb{E}(\sqrt{\mathcal{V}})}{48} \left( \mathbb{E}(\mathcal{V}^{3/2}) - (\mathbb{E}(\sqrt{\mathcal{V}}))^3 \right) \tau,$$

The small-maturity convexity follows similar arguments, which we only outline:

$$(2.15) \quad \begin{aligned} \partial_{kk} C(k, \tau) &= \partial_{kk} \text{BS}(k, \sigma_\tau^2(k), \tau) + 2\partial_k \sigma_\tau^2(k) \partial_{wk} \text{BS}(k, \sigma_\tau^2(k), \tau) + (\partial_k \sigma_\tau^2(k))^2 \partial_{ww} \text{BS}(k, \sigma_\tau^2(k), \tau) \\ &\quad + \partial_{kk} \sigma_\tau^2(k) \partial_w \text{BS}(k, \sigma_\tau^2(k), \tau), \end{aligned}$$

and

$$(2.16) \quad \partial_{kk} C(k, \tau) = \int_0^\infty \partial_{kk} \text{BS}(k, y, \tau) \zeta_p(y) dy + m_t \partial_{kk} (1 - e^k)^+.$$

Likewise, we first consider the case  $m_t = 0$ . Straightforward computations yield

$$(2.17) \quad \begin{cases} \partial_{kk} \text{BS}(k, \sigma_\tau^2(k), \tau)|_{k=0} = \frac{\exp\left(-\frac{\sigma_\tau^2(0)\tau}{8}\right)}{\sigma_\tau(0)\sqrt{\tau}\sqrt{2\pi}} - \mathcal{N}\left(\frac{\sigma_\tau(0)\sqrt{\tau}}{2}\right) = \frac{1}{\sqrt{2\pi}\sigma_\tau(0)\sqrt{\tau}} - \frac{1}{2} + \mathcal{O}(\sigma_\tau(0)\sqrt{\tau}), \\ \partial_{kw} \text{BS}(k, \sigma_\tau^2(k), \tau)|_{k=0} = \exp\left(-\frac{\sigma_\tau^2(0)\tau}{8}\right) \frac{\sqrt{\tau}}{4\sigma_\tau(0)\sqrt{2\pi}}, \\ \partial_{ww} \text{BS}(k, \sigma_\tau^2(k), \tau)|_{k=0} = -\frac{\tau^2 e^{-\frac{\tau\sigma_\tau^2(0)}{8}} (\tau\sigma_\tau^2(0) + 4)}{16\sqrt{2\pi}(\tau\sigma_\tau^2(0))^{3/2}} = -\frac{\sqrt{\tau}}{4\sqrt{2\pi}\sigma_\tau^3(0)} + \frac{3\tau^{5/2}\sigma_\tau(0)}{512\sqrt{2\pi}} + \mathcal{O}\left(\frac{\tau^{3/2}}{\sigma_\tau(0)}\right). \end{cases}$$

Using (2.15) and (2.16) in conjunction with (2.13), (2.17) and (2.14), we obtain  $\partial_{kk} \sigma_\tau^2(0) \sim \frac{1}{\tau} \mathbb{E}(\sqrt{\mathcal{V}}) \left( \mathbb{E}(\mathcal{V}^{-1/2}) - \mathbb{E}(\sqrt{\mathcal{V}})^{-1} \right)$ . When  $m_t > 0$ , we need to take right and left derivatives in (2.12) and (2.16) to account for the atomic term. Since  $\partial_k^- (1 - e^k)^+|_{k=0} = \partial_{kk}^- (1 - e^k)^+|_{k=0} = -1$  and  $\partial_k^+ (1 - e^k)^+|_{k=0} = \partial_{kk}^+ (1 - e^k)^+|_{k=0} = 0$ , the lemma follows immediately.  $\square$

**2.5. Large-time behaviour of option prices and implied volatility.** In this section we compute the large-time behaviour of option prices and implied volatility. The proofs are given in Section 3.2. It turns out that asymptotics are degenerate in the sense that option prices decay algebraically to their intrinsic values. The structure of the asymptotic depends on the CEV parameter  $p$  and whether the origin is reflecting or absorbing:

**Theorem 2.8.** *Define the following quantity:*

$$\mathfrak{M}(\eta) := \frac{2^{3-6p-\eta} \Gamma\left(\frac{1}{2} - 2p\right)}{\sqrt{\pi} \Gamma(1 + \eta) |1 - p|^{2\eta+1} (\xi^2 t)^{\eta+1}} \exp\left(-\frac{y_0^{2(1-p)}}{2\xi^2 t(1-p)^2}\right),$$

with  $\eta$  given in (2.4). The following expansions hold for all  $k \in \mathbb{R}$  as  $\tau$  tends to infinity:

(i) if  $p < 3/4$  and the origin is absorbing then

$$\mathbb{E}(e^{Z_\tau} - e^k)^+ = 1 - m_t + m_t(1 - e^k)^+ - 8e^{k/2} y_0 \left(\frac{1}{2} - 2p\right) \mathfrak{M}(-\eta) \frac{1 + \mathcal{O}(\tau^{-1})}{\tau^{2-2p}};$$

(ii) if  $p < 1/4$  and the origin is reflecting then

$$\mathbb{E}(e^{Z_\tau} - e^k)^+ = 1 - e^{k/2} \mathfrak{M}(\eta) \frac{1 + \mathcal{O}(\tau^{-1})}{\tau^{1-2p}}.$$

For other values of  $p$ , asymptotics are more difficult to derive and we leave this for future research. The asymptotic behaviour of option prices is fundamentally different to Black-Scholes asymptotics (Lemma A.2) and it is not clear that one can deduce asymptotics for the implied volatility. For example, the intrinsic values do not necessarily match as  $\tau$  tends to infinity because of the mass at the origin. The one exception is when the origin is reflecting, in which case the implied volatility tends to zero. This result follows directly from the comparison of Theorem 2.8 and Lemma A.2.

**Theorem 2.9.** *If  $p < 1/4$  and the origin is reflecting, then for all  $k \in \mathbb{R}$  as  $\tau$  tends to infinity:*

$$\sigma_\tau^2(k) \sim \frac{8(1-2p) \log \tau}{\tau}.$$

Although, we have provided the large-time asymptotics in this section, it is not our intention to use this model for options with large expiries. Our intention (as mentioned in Section 1) is to use these models as building blocks for more complicated models (such as stochastic volatility models where the initial variance is sampled from a continuous distribution) so that we are able to better match steep small-maturity observed smiles. In these types of more sophisticated models, the large-time behaviour is governed more from the chosen stochastic volatility model rather than the choice of distribution for the initial variance (see [39, 40] for examples), especially if the variance process possesses some ergodic properties. This also suggests to use this class of models to introduce two different time scales: one to match the small-time smile (the distribution for the initial variance) and one to match the medium- to large-time smile (the chosen stochastic volatility model).

**2.6. Numerics.** We calculate option prices using the representation (3.1) and a global adaptive Gauss-Kronrod quadrature scheme. We then compute the smile with a simple root-finding algorithm. In Figure 1 we plot the smile for different maturities and values for the CEV power  $p$ . The model parameters are  $y_0 = 0.07$ ,  $\xi = 0.2y_0^{1/2-p}$  and  $t = 1/2$ . Note here that we set  $\xi$  to be a different value for each  $p$ . This is done so that the models are comparable:  $\xi$  is then given in the same units and the quadratic variation of the CEV variance dynamics are approximately matched for different values of  $p$ . The graphs highlight the steepness of the smiles as the maturity gets smaller and the role of  $p$  in the shape of the small-maturity smile. Out-of-the money volatilities (for  $K \notin [0.9, 1.1]$ ) explode at a quicker rate as  $p$  increases (this can be seen from Theorem 2.3). The volatility for strikes close to the money  $K \in [0.9, 1.1]$  appears to be less explosive as one increases  $p$ . This can be explained from the strike dependence of the coefficients of the asymptotic in Theorem 2.3 and is discussed further in Section 2.7.1.

**2.7. Application to forward smile asymptotics.** We now show how our model (2.1) and the asymptotics derived above for the implied volatility can be directly translated into asymptotics of the forward implied volatility in stochastic volatility models. For a given martingale process  $e^X$ , a forward-start option with reset date  $t$ , maturity  $t + \tau$  and strike  $e^k$  is worth, at inception,  $\mathbb{E}(\exp(X_{t+\tau} - X_t) - e^k)_+$ . In the Black-Scholes model, the stationarity property of the increments imply that this option is simply equal to a standard Call option on  $e^X$  (started at  $X_0 = 0$ ) with strike  $e^k$  and maturity  $\tau$ ; therefore, one can define the forward implied volatility  $\sigma_{t,\tau}(k)$ , similarly to the standard implied volatility (see [39] for more details). Suppose now that the log stock price process  $X$  satisfies the following SDE:

$$(2.18) \quad \begin{aligned} dX_s &= -\frac{1}{2}Y_s ds + \sqrt{Y_s} dW_s, & X_0 &= 0, \\ dY_s &= \xi_s Y_s^p dB_s, & Y_0 &= y_0 > 0, \\ d\langle W, B \rangle_s &= \rho ds, \end{aligned}$$

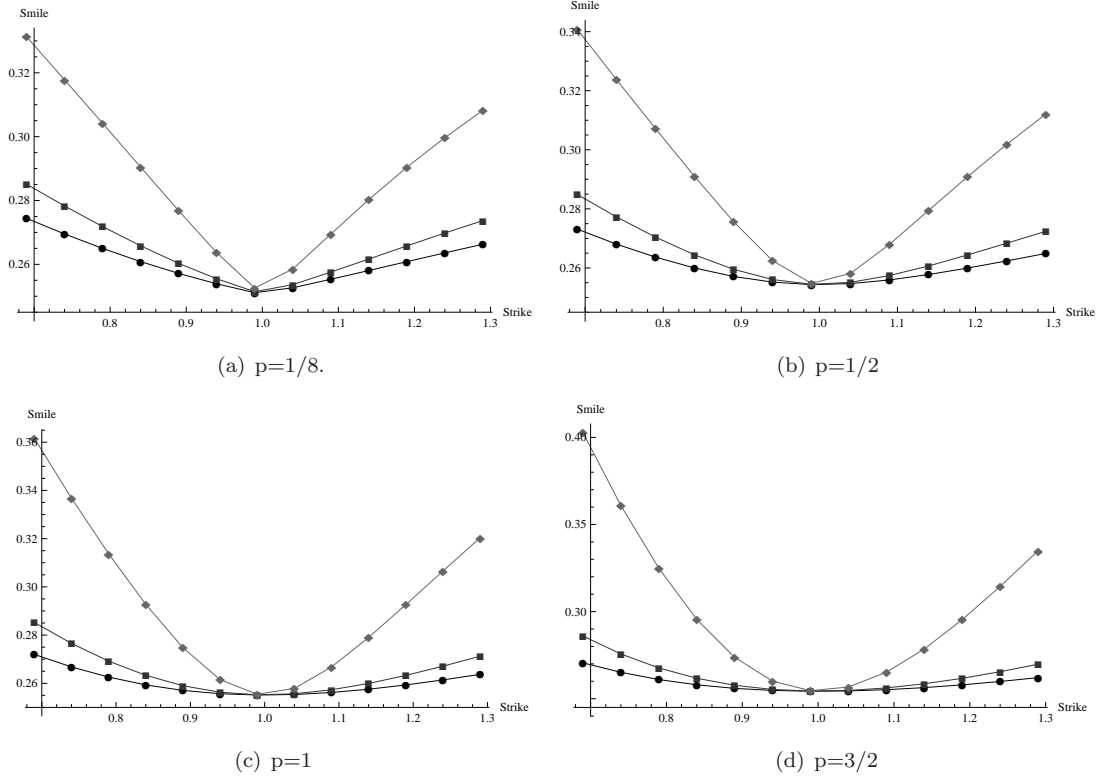


FIGURE 1. Here we plot  $K \mapsto \sigma_\tau(\log K)$  for maturities of 1 (circles),  $1/2$  (squares) and  $1/12$  (diamonds) for increasing values of the CEV power  $p$ . The smile is obtained by numerically solving for the option price using (3.1) and then using a simple root search to solve for the implied volatility. Parameters of the model are given in the text.

with  $p \in \mathbb{R}$ ,  $|\rho| < 1$  and  $W, B$  are two standard Brownian motions. Fix the forward-start date  $t > 0$  and set

$$(2.19) \quad \xi_u := \begin{cases} \xi, & \text{if } 0 \leq u \leq t, \\ \bar{\xi}, & \text{if } u > t, \end{cases}$$

where  $\xi > 0$  and  $\bar{\xi} \geq 0$ . This includes the Heston model and 3/2 model with zero mean reversion ( $p = 1/2$  and  $p = 3/2$  respectively) as well as the SABR model ( $p = 1$ ). Consider the CEV process for the variance:  $dY_u = \xi Y_u^p dB_u$ ,  $Y_0 = y_0$ , where  $p \in \mathbb{R}$  and  $B$  is a standard Brownian motion. Let  $X_\tau^{(t)} := X_{t+\tau} - X_\tau$  denote the forward price process and let  $\text{CEV}(t, \xi, p)$  be the distribution such that  $\text{Law}(Y_t) = \text{Law}(\mathcal{V}) = \text{CEV}(t, \xi, p)$ . Then the following lemma holds:

**Lemma 2.10.** *In the model (2.18) the forward price process  $X_\tau^{(t)}$  solves the following system of SDEs:*

$$(2.20) \quad \begin{aligned} dX_\tau^{(t)} &= -\frac{1}{2}Y_\tau^{(t)}d\tau + \sqrt{Y_\tau^{(t)}}dW_\tau, & X_0^{(t)} &= 0, \\ dY_\tau^{(t)} &= \bar{\xi} \left(Y_\tau^{(t)}\right)^p dB_\tau, & Y_0^{(t)} &\sim \text{CEV}(t, \xi, p), \\ d\langle W, B \rangle_\tau &= \rho d\tau, \end{aligned}$$

where  $Y_0^{(t)}$  is independent to the Brownian motions  $(W_\tau)_{\tau \geq 0}$  and  $(B_\tau)_{\tau \geq 0}$ .

This lemma makes it clear that forward-start options in stochastic volatility models are European options on a stock price with similar dynamics to (2.18), but with initial variance sampled from the variance distribution at the forward-start date. If we set  $\bar{\xi} = 0$ , then  $X^{(t)} = Z$  and the following corollary provides forward smile asymptotics:

**Corollary 2.11.** *If  $\bar{\xi} = 0$ , Theorem 2.1, Theorem 2.3 and Lemma 2.4 hold with  $Z = X^{(t)}$  and  $\sigma_\tau = \sigma_{t,\tau}$ .*

**Remark 2.12.**

- (i) This result explicitly links the shape and fatness of the right tail of the variance distribution at the forward-start date and the asymptotic form and explosion rate of the small-maturity forward smile. Take for example  $p > 1$ : the density of the variance in the right wing is dominated by the polynomial  $y^{-2p}$  and the exponential dependence on  $y$  is irrelevant. So the smaller  $p$  in this case, the fatter the right tail and hence the larger the coefficient of the expansion. This also explains the algebraic (not exponential) dependence for forward-start option prices.
- (ii) The asymptotics in the  $p > 1$  case are extreme and the algebraic dependence on  $\tau$  is similar to small-maturity exponential Lévy models. This extreme nature is related to the fatness of the right tail of the variance distribution: for example, the 3/2 model ( $p = 3/2$ ) allows for the occurrence of extreme paths with periods of very high instantaneous volatility (see [22, Figure 3]).
- (iii) The asymptotics in Theorems 2.1 and 2.3 remain the same (at this order) regardless of whether the variance process is absorbing or reflecting at zero when  $p \in (-\infty, 1/2)$ . Intuitively this is because absorption or reflection primarily influences the left tail whereas small-maturity forward smile asymptotics are influenced by the shape of the right tail of the variance distribution.

**2.7.1. Conjecture.** When  $p = 1/2$  in Corollary 2.11, the asymptotics are the same as in [41, Theorem 4.1] for the Heston model. This confirms that the key quantity determining the small-maturity forward smile explosion rate is the variance distribution at the forward-start date. The dynamics of the stock price are actually irrelevant at this order. This leads us to the following conjecture:

**Conjecture 2.13.** *The leading-order small-maturity forward smile asymptotics generated from (2.18) are equivalent to those given in Corollary 2.11.*

Practitioners have stated [4, 12] that the Heston model ( $p = 1/2$ ) produces small-maturity forward smiles that are too convex and "U-shaped" and inconsistent with observations. Furthermore, it has been empirically stated [4] that SABR or lognormal based models for the variance ( $p = 1$ ) produce less convex or "U-shaped" small-maturity forward smiles. Our results provide theoretical insight into this effect. We observed in Section 2.6 and Figure 1 that the explosion effect was more stable for strikes close to the money as one increased  $p$ . The strike dependence of the asymptotic implied volatility in Theorem 2.3 is given by  $K \mapsto \sqrt{|\log K|}$  for  $p = 1/2$  and  $K \mapsto |\log K|$  for  $p = 1$ . It is clear that the shape of the forward implied volatility is more stable and less U-shaped in the lognormal  $p = 1$  case.

### 3. PROOFS

**3.1. Proof of Theorem 2.1.** Let  $C(k, \tau) := \mathbb{E}(e^{X_\tau} - e^k)^+$ . This function clearly depends on the parameter  $t$ , but we omit this dependence in the notations. The tower property implies

$$(3.1) \quad C(k, \tau) = \int_0^\infty \text{BS}(k, y, \tau) \zeta_p(y) dy + m_t (1 - e^k)^+,$$

where BS is defined in (2.8),  $\zeta_p$  is density of  $\mathcal{V}$  given in (2.5) and  $m_t$  is the mass at the origin (2.3). Our goal is to understand the asymptotics of this integral as  $\tau$  tends to zero. We break the proof of Theorem 2.1 into three parts: in Section 3.1.1 we prove the case  $p > 1$ , in Section 3.1.2 we prove the case  $p \in (-\infty, 1)$  and in Section 3.1.3 we prove the case  $p = 1$ . We only prove the result for  $k > 0$ , the arguments being completely analogous when  $k < 0$ . The key insight is that one has to re-scale the variance in terms of the maturity  $\tau$  before asymptotics can be computed. The nature of the re-scaling depends critically on the CEV power  $p$  and fundamentally different asymptotics result in each case. Note that for  $k > 0$ ,  $(1 - e^k)^+ = 0$ , so that the atomic term in (3.1) is irrelevant for the analysis. When  $k < 0$  the arguments are analogous by Put-Call parity.

**3.1.1. Case:  $p > 1$ .** In Lemma 3.1 we prove a bound on the CEV density. This is sufficient to allow us to prove asymptotics for option prices in Lemma 3.2 after rescaling the variance by  $\tau$ . This rescaling is critical because it is the only one making  $\text{BS}(k, y/\tau, \tau)$  independent of  $\tau$ . Let

$$\chi(\tau, p) := \frac{\tau^{2p}}{|1 - p|\xi^2 t \Gamma(1 + |\eta|) (2(1 - p)^2 \xi^2 t)^{|\eta|}} \exp\left(-\frac{y_0^{2(1-p)}}{2\xi^2 t(1-p)^2}\right),$$

and we have the following lemma:

**Lemma 3.1.** *The following bounds hold for the CEV density for all  $y, \tau > 0$  when  $p > 1$ :*

$$\begin{aligned} \zeta_p\left(\frac{y}{\tau}\right) &\geq \frac{\chi(\tau, p)}{y^{2p}} \left\{ 1 - \frac{1}{2\xi^2 t(1-p)^2} \left(\frac{\tau}{y}\right)^{2p-2} \right\}, \\ \zeta_p\left(\frac{y}{\tau}\right) &\leq \frac{\chi(\tau, p)}{y^{2p}} \left\{ 1 + \exp\left(\frac{y_0^{2-2p}}{2(p-1)^2 t \xi^2}\right) \left[ \frac{1}{2\xi^2 t(1-p)^2} \left(\frac{\tau}{y}\right)^{2p-2} + \frac{1}{\xi^2 t(1-p)^2} \left(\frac{\tau}{yy_0}\right)^{p-1} \right] \right\}. \end{aligned}$$

*Proof.* From [47, Equation (6.25)] we know that for  $x > 0$  and  $\nu > -1/2$ , the modified Bessel function satisfies

$$(3.2) \quad \frac{1}{\Gamma(\nu+1)} \left(\frac{x}{2}\right)^\nu \leq I_\nu(x) \leq \frac{e^x}{\Gamma(\nu+1)} \left(\frac{x}{2}\right)^\nu,$$

so that the expression for the CEV density in (2.5) implies that for  $p > 1$ ,

$$\frac{\chi(\tau, p)}{y^{2p}} \exp\left(-\frac{1}{2\xi^2 t(1-p)^2} \left(\frac{\tau}{y}\right)^{2p-2}\right) \leq \zeta_p\left(\frac{y}{\tau}\right) \leq \frac{\chi(\tau, p)}{y^{2p}} e^{m(y, \tau)},$$

where

$$m(y, \tau) := -\frac{1}{2\xi^2 t(1-p)^2} \left(\frac{\tau}{y}\right)^{2p-2} + \frac{1}{\xi^2 t(1-p)^2} \left(\frac{\tau}{yy_0}\right)^{p-1}.$$

For fixed  $\tau > 0$ , note that  $m(\cdot, \tau) : \mathbb{R}_+ \mapsto \mathbb{R}_+$  takes a maximum positive value at  $y = y_0\tau$  with  $m(y_0\tau, \tau) = y_0^{2-2p}/(2(p-1)^2 t \xi^2)$ . When  $m(\cdot) > 0$  Taylor's Theorem with remainder yields  $e^{m(y, \tau)} = 1 + e^\gamma m(y, \tau)$  for some  $\gamma \in (0, m(y, \tau))$ , and hence  $e^{m(y, \tau)} \leq 1 + e^{m(y_0\tau, \tau)} m(y, \tau)$ . If  $m(\cdot) < 0$  then  $e^{m(y, \tau)} \leq 1 + |m(y, \tau)| \leq 1 + e^{m(y_0\tau, \tau)} |m(y, \tau)|$ . The result for the upper bound then follows by the triangle inequality for  $|m(y, \tau)|$ . The lower bound simply follows from the inequality  $1 - x \leq e^{-x}$ , valid for  $x > 0$ , and

$$1 - \frac{1}{2\xi^2 t(1-p)^2} \left(\frac{\tau}{y}\right)^{2p-2} \leq \exp\left(-\frac{1}{2\xi^2 t(1-p)^2} \left(\frac{\tau}{y}\right)^{2p-2}\right).$$

□

**Lemma 3.2.** *When  $p > 1$ , Theorem 2.1 holds.*



*Proof.* The substitution  $y \rightarrow y/\tau$  and (3.1) imply that the option price reads  $C(k, \tau) = \int_0^\infty \text{BS}(k, y, \tau) \zeta_p(y) dy = \tau^{-1} \int_0^\infty \text{BS}(k, y/\tau, \tau) \zeta_p(y/\tau) dy$ . Using Lemma 3.1 and Definition (2.9), we obtain the following bounds:

$$\begin{aligned} \frac{\chi(\tau, p)}{\tau} \left[ J^{2p}(k) - \frac{\tau^{2p-2}}{2\xi^2 t(1-p)^2} J^{4p-2}(k) \right] &\leq C(k, \tau), \\ \frac{\chi(\tau, p)}{\tau} \left[ J^{2p}(k) + \exp\left(\frac{y_0^{2-2p}}{2(p-1)^2 t \xi^2}\right) \left( \frac{\tau^{2p-2}}{2\xi^2 t(1-p)^2} J^{4p-2}(k) + \frac{\tau^{p-1}}{\xi^2 t(1-p)^2 y_0^{p-1}} J^{3p-1}(k) \right) \right] &\geq C(k, \tau). \end{aligned}$$

Hence for  $\tau < 1$ :

$$\left| \frac{C(k, \tau) \tau}{\chi(\tau, p) J^{2p}(k)} - 1 \right| \leq \exp\left(\frac{y_0^{2-2p}}{2(p-1)^2 t \xi^2}\right) \left( \frac{J^{4p-2}(k)}{2\xi^2 t(1-p)^2 J^{2p}(k)} + \frac{J^{3p-1}(k)}{\xi^2 t(1-p)^2 y_0^{p-1} J^{2p}(k)} \right) \tau^{p-1},$$

which proves the lemma since  $J^{2p}(k)$  is strictly positive, finite and independent of  $\tau$ .  $\square$

**3.1.2. Case:  $p < 1$ .** We use the representation in (3.1) and break the domain of the integral up into a compact part and an infinite (tail) one. We prove in Lemma 3.4 that the tail integral is exponentially sub-dominant (compared to the compact part) and derive asymptotics for the integral in Lemma 3.5. This allows us to apply the Laplace method to the integral. We start with the following bound for the modified Bessel function of the first kind and then prove a tail estimate in Lemma 3.4.

**Lemma 3.3.** *The following bound holds for all  $x > 0$  and  $\nu > -3/2$ :*

$$I_\nu(x) < \frac{\nu+2}{\Gamma(\nu+2)} \left(\frac{x}{2}\right)^\nu e^{2x}.$$

*Proof.* Let  $x > 0$ . From [55, Theorem 7, page 522], the inequality  $I_\nu(x) < I_{\nu+1}(x)^2 / I_{\nu+2}(x)$  holds whenever  $\nu \geq -2$ , and hence combining it with (3.2) (valid only for  $\nu > -1/2$ ), we can write

$$I_\nu(x) < \frac{\Gamma(\nu+3)}{\Gamma(\nu+2)^2} \left(\frac{x}{2}\right)^\nu e^{2x},$$

when  $\nu > -3/2$ . The lemma then follows from the trivial identity  $\Gamma(\nu+3) = (\nu+2)\Gamma(\nu+2)$ .  $\square$

**Lemma 3.4.** *Let  $L > 1$  and  $p < 1$ . Then the following tail estimate holds as  $\tau$  tends to zero:*

$$\int_L^\infty \text{BS}\left(k, \frac{y}{\tau^{\beta_p}}, \tau\right) \zeta_p\left(\frac{y}{\tau^{\beta_p}}\right) dy = \mathcal{O}\left(\exp\left(-\frac{1}{4\xi^2 t(1-p)} \left[\frac{L^{1-p}}{\tau^{(1-\beta_p)/2}} - y_0^{1-p}\right]^2\right)\right).$$

*Proof.* Lemma 3.3 and the density in (2.5) imply

$$\zeta_p\left(\frac{y}{\tau^{\beta_p}}\right) \leq \frac{b_0}{\tau^{-2p\beta_p}} y^{-2p} \exp\left(-\frac{1}{2\xi^2 t(1-p)^2} \left\{ \frac{y^{1-p}}{\tau^{\beta_p(1-p)}} - y_0^{1-p} \right\}^2 + \frac{(yy_0)^{1-p}}{\tau^{\beta_p(1-p)} \xi^2 t(1-p)^2}\right),$$

where the constant  $b_0$  is given by

$$\frac{(\eta+2)}{|1-p|\xi^2 t \Gamma(\eta+2) (2(1-p)^2 \xi^2 t)^\eta}, \quad \text{resp.} \quad \frac{(|\eta|+2)}{|1-p|\xi^2 t \Gamma(|\eta|+2) (2(1-p)^2 \xi^2 t)^{|\eta|}},$$

if the origin is reflecting (resp. absorbing) when  $p < 1/2$ ; the exact value of  $b_0$  is however irrelevant for the analysis. Set now  $L > 1$ . Using this upper bound and the no-arbitrage inequality  $\text{BS}(\cdot) \leq 1$ , we find

$$\begin{aligned} \int_L^\infty \text{BS}\left(k, \frac{y}{\tau^{\beta_p}}, \tau\right) \zeta_p\left(\frac{y}{\tau^{\beta_p}}\right) dy &\leq \int_L^\infty \zeta_p\left(\frac{y}{\tau^{\beta_p}}\right) dy \\ &\leq \frac{b_0}{\tau^{-2p\beta_p}} \int_L^\infty y^{-2p} \exp\left(-\frac{1}{2\xi^2 t(1-p)^2} \left\{\frac{y^{1-p}}{\tau^{\beta_p(1-p)}} - y_0^{1-p}\right\}^2 + \frac{(yy_0)^{1-p}}{\tau^{\beta_p(1-p)}\xi^2 t(1-p)^2}\right) dy \\ &\leq \frac{b_0}{\tau^{-2p\beta_p}} \int_L^\infty y^{1-2p} \exp\left(-\frac{1}{2\xi^2 t(1-p)^2} \left\{\frac{y^{1-p}}{\tau^{\beta_p(1-p)}} - y_0^{1-p}\right\}^2 + \frac{(yy_0)^{1-p}}{\tau^{\beta_p(1-p)}\xi^2 t(1-p)^2}\right) dy, \end{aligned}$$

where the last line follows since  $y^{1-2p} > y^{-2p}$ . Setting  $q = \left(y^{1-p}/\tau^{\beta_p(1-p)} - y_0^{1-p}\right)/(\xi\sqrt{t}(1-p))$  yields

$$\begin{aligned} &\int_L^\infty y^{1-2p} \exp\left(-\frac{\left(\frac{y^{1-p}}{\tau^{\beta_p(1-p)}} - y_0^{1-p}\right)^2}{2\xi^2 t(1-p)^2} + \frac{(yy_0)^{1-p}}{\tau^{\beta_p(1-p)}\xi^2 t(1-p)^2}\right) dy \\ (3.3) \quad &= \frac{\xi\sqrt{t}(1-p)}{\tau^{2\beta_p(p-1)}} \left[ \xi\sqrt{t}(1-p) \int_{L_\tau}^\infty q \exp\left[-\frac{q^2}{2} + \frac{y_0^{1-p}q}{\xi\sqrt{t}(1-p)}\right] dq + y_0^{1-p} \int_{L_\tau}^\infty \exp\left[-\frac{q^2}{2} + \frac{y_0^{1-p}q}{\xi\sqrt{t}(1-p)}\right] dq \right], \end{aligned}$$

with  $L_\tau := \left(L^{1-p}/\tau^{\beta_p(1-p)} - y_0^{1-p}\right)/(\xi\sqrt{t}(1-p)) > 0$  for small enough  $\tau$  since  $L > 1$  and  $p \in (-\infty, 1)$ . Set now (we always choose the positive root)

$$\tau^* := \left(\frac{L^{1-p}}{5y_0^{1-p}}\right)^{(\beta_p(1-p))^{-1}},$$

so that, for  $\tau < \tau^*$  we have  $L_\tau > 4y_0^{1-p}/(\xi\sqrt{t}(1-p))$  and hence for  $q > L_\tau$ :

$$\frac{y_0^{1-p}q}{\xi\sqrt{t}(1-p)} \leq \frac{q^2}{4}.$$

In particular, for the integrals in (3.3) we have the following bounds for  $\tau < \tau^*$ :

$$\begin{aligned} \int_{L_\tau}^\infty q \exp\left(-\frac{q^2}{2} + \frac{y_0^{1-p}q}{\xi\sqrt{t}(1-p)}\right) dq &\leq \int_{L_\tau}^\infty q \exp\left(-\frac{q^2}{4}\right) dq = 2 \exp\left(-\frac{L_\tau^2}{4}\right), \\ \int_{L_\tau}^\infty \exp\left(-\frac{q^2}{2} + \frac{y_0^{1-p}q}{\xi\sqrt{t}(1-p)}\right) dq &\leq \int_{L_\tau}^\infty \exp\left(-\frac{q^2}{4}\right) dq \leq \frac{4}{L_\tau} \exp\left(-\frac{L_\tau^2}{4}\right), \end{aligned}$$

where the last inequality follows from the upper bound for the complementary normal distribution function in [59, Section 14.8]. The lemma then follows from noting that  $1 - \beta_p = 2\beta_p(1-p)$ .  $\square$

**Lemma 3.5.** *When  $p < 1$ , Theorem 2.1 holds.*

*Proof.* Let  $\tilde{\tau} := \tau^{\beta_p}$ , with  $\beta_p$  defined in (2.10). Applying the substitution  $y \rightarrow y/\tilde{\tau}$  to (3.1) yields

$$\begin{aligned} C(k, \tau) &= \int_0^\infty \text{BS}(k, y, \tau) \zeta_p(y) dy = \frac{1}{\tilde{\tau}} \int_0^\infty \text{BS}\left(k, \frac{y}{\tilde{\tau}}, \tau\right) \zeta_p\left(\frac{y}{\tilde{\tau}}\right) dy \\ &= \frac{1}{\tilde{\tau}} \int_0^L \text{BS}\left(k, \frac{y}{\tilde{\tau}}, \tau\right) \zeta_p\left(\frac{y}{\tilde{\tau}}\right) dy + \frac{1}{\tilde{\tau}} \int_L^\infty \text{BS}\left(k, \frac{y}{\tilde{\tau}}, \tau\right) \zeta_p\left(\frac{y}{\tilde{\tau}}\right) dy, \end{aligned}$$

for some  $L > 0$  to be chosen later. We start with the first integral. Using the asymptotics for the modified Bessel function of the first kind [1, Section 9.7.1] as  $\tau$  tends to zero, we obtain

$$\zeta_p\left(\frac{y}{\tau}\right) = \frac{\tau^{3p\beta_p/2} y_0^{p/2} e^{-\frac{y_0^{2(1-p)}}{2\xi^2 t(1-p)^2}}}{\xi y^{3p/2} \sqrt{2\pi t}} \exp\left(-\frac{1}{\tau^{2\beta_p(1-p)}} \frac{y^{2(1-p)}}{2\xi^2 t(1-p)^2} + \frac{1}{\tau^{\beta_p(1-p)}} \frac{(yy_0)^{(1-p)}}{\xi^2 t(1-p)^2}\right) \left[1 + \mathcal{O}\left(\tau^{(1-p)\beta_p}\right)\right].$$

Note that this expansion does not depend on the sign of  $\eta$  and so the same asymptotics hold regardless of whether the origin is reflecting or absorbing. In the Black-Scholes model, Call option prices satisfy (Lemma A.1):

$$\text{BS}\left(k, \frac{y}{\tau}, \tau\right) = \frac{y^{3/2}}{k^2 \sqrt{2\pi}} \left(\frac{\tau}{\tilde{\tau}}\right)^{3/2} \exp\left(-\frac{k^2 \tilde{\tau}}{2y\tau} + \frac{k}{2}\right) \left(1 + \mathcal{O}\left(\frac{\tau}{\tilde{\tau}}\right)\right),$$

as  $\tau$  tends to zero. Using the identity  $1 - \beta_p = 2\beta_p(1-p)$  we then compute

$$\begin{aligned} \frac{1}{\tau^{\beta_p}} \int_0^L \text{BS}\left(k, \frac{y}{\tau^{\beta_p}}, \tau\right) \zeta_p\left(\frac{y}{\tau^{\beta_p}}\right) dy \\ = \frac{\tau^{\beta_p(4-3p)/2} y_0^{p/2} e^{-\frac{y_0^{2(1-p)}}{2\xi^2 t(1-p)^2} + \frac{k}{2}}}{2\pi k^2 \xi \sqrt{t}} \int_0^L y^{\frac{3}{2}(1-p)} e^{-\frac{f_0(y)}{\tau^{1-\beta_p}} + \frac{f_1(y)}{\tau^{(1-\beta_p)/2}}} dy \left[1 + \mathcal{O}\left(\tau^{(1-\beta_p)/2}\right)\right], \end{aligned}$$

where  $f_0, f_1$  are defined in (2.11). Solving the equation  $f'_0(y) = 0$  gives  $y = \bar{y}_p$  with  $\bar{y}_p$  defined in (2.10) and we always choose the positive root and set  $L > \bar{y}_p$ . Let  $I(\tau) := \int_0^L y^{\frac{3}{2}(1-p)} \exp\left(-\frac{f_0(y)}{\tau^{1-\beta_p}} + \frac{f_1(y)}{\tau^{(1-\beta_p)/2}}\right) dy$ . Then for some  $\varepsilon > 0$  small enough, as  $\tau$  tends to zero:

$$\begin{aligned} I(\tau) &\sim \exp\left(-\frac{f_0(\bar{y}_p)}{\tau^{1-\beta_p}} + \frac{f_1(\bar{y}_p)}{\tau^{(1-\beta_p)/2}} + \frac{f'_1(\bar{y}_p)^2}{2f''_0(\bar{y}_p)}\right) \bar{y}_p^{\frac{3}{2}(1-p)} \int_{\bar{y}_p-\varepsilon}^{\bar{y}_p+\varepsilon} \exp\left(-\frac{1}{2} \left[\frac{\sqrt{f''_0(\bar{y}_p)}(y-\bar{y}_p)}{\tau^{(1-\beta_p)/2}} - \frac{f'_1(\bar{y}_p)}{\sqrt{f''_0(\bar{y}_p)}}\right]^2\right) dy \\ &\sim \exp\left(-\frac{f_0(\bar{y}_p)}{\tau^{1-\beta_p}} + \frac{f_1(\bar{y}_p)}{\tau^{(1-\beta_p)/2}} + \frac{f'_1(\bar{y}_p)^2}{2f''_0(\bar{y}_p)}\right) \bar{y}_p^{\frac{3}{2}(1-p)} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2} \left[\frac{\sqrt{f''_0(\bar{y}_p)}(y-\bar{y}_p)}{\tau^{(1-\beta_p)/2}} - \frac{f'_1(\bar{y}_p)}{\sqrt{f''_0(\bar{y}_p)}}\right]^2\right) dy \\ &= \exp\left(-\frac{f_0(\bar{y}_p)}{\tau^{1-\beta_p}} + \frac{f_1(\bar{y}_p)}{\tau^{(1-\beta_p)/2}} + \frac{f'_1(\bar{y}_p)^2}{2f''_0(\bar{y}_p)}\right) \tau^{(1-\beta_p)/2} \bar{y}_p^{\frac{3}{2}(1-p)} \sqrt{\frac{2\pi}{f''_0(\bar{y}_p)}}. \end{aligned}$$

The  $\sim$  approximations here are exactly of the same type as in [32], and we refer the interested reader to this paper for details. It follows that as  $\tau$  tends to zero:

$$\frac{1}{\tau^{\beta_p}} \int_0^L \text{BS}\left(k, \frac{y}{\tau^{\beta_p}}, \tau\right) \zeta_p\left(\frac{y}{\tau^{\beta_p}}\right) dy = \exp\left(-\frac{c_1(t, p)}{\tau^{1-\beta_p}} + \frac{c_2(t, p)}{\tau^{(1-\beta_p)/2}}\right) c_5(t, p) \tau^{c_3(t, p)} \left[1 + \mathcal{O}\left(\tau^{\frac{1-\beta_p}{2}}\right)\right].$$

From Lemma 3.4 we know that

$$\frac{1}{\tau^{\beta_p}} \int_L^\infty \text{BS}\left(k, \frac{y}{\tau^{\beta_p}}, \tau\right) \zeta_p(y/\beta_p) dy = \mathcal{O}\left(\exp\left(-\frac{1}{2\xi^2 t(1-p)} \left(\frac{L^{1-p}}{\tau^{(1-\beta_p)/2}} - y_0^{1-p}\right)^2\right)\right).$$

Choosing  $L > \max\left(1, (2\xi^2 t(1-p)f_0(\bar{y}_p))^{1/(2-2p)}, \bar{y}_p\right)$  makes this tail term exponentially subdominant to  $\tau^{-\beta_p} \int_0^L \text{BS}(k, y/\tau^{\beta_p}, \tau) \zeta_p(y/\beta_p) dy$ , which completes the proof of the lemma.  $\square$

3.1.3. *Case:  $p = 1$ .* We now consider the lognormal case  $p = 1$ . In this case, the random variable  $\log(\mathcal{V})$  is Gaussian with mean  $\mu$  (defined in (2.4)) and variance  $\xi^2 t$ . The proof is similar to Section 3.1.2, but we need to re-scale the variance by  $\tau |\log(\tau)|$ . We prove a tail estimate in Lemma 3.6 and derive asymptotics for option prices in Lemma 3.7.

**Lemma 3.6.** *The following tail estimate holds for  $p = 1$  and  $L > 0$  as  $\tau$  tends to zero ( $\mu$  defined in (2.4)):*

$$\int_L^\infty \text{BS}\left(k, \frac{y}{\tau |\log(\tau)|}, \tau\right) \zeta_1\left(\frac{y}{\tau |\log(\tau)|}\right) dy = \mathcal{O}\left(\exp\left\{-\frac{1}{2\xi^2 t} \left[\log\left(\frac{L}{\tau |\log(\tau)|}\right) - \mu\right]^2\right\}\right).$$

*Proof.* By no-arbitrage arguments, the Call price is always bounded above by one, so that

$$\int_L^\infty \text{BS}\left(k, \frac{y}{\tau |\log(\tau)|}, \tau\right) \zeta_1\left(\frac{y}{\tau |\log(\tau)|}\right) dy \leq \int_L^\infty \zeta_1\left(\frac{y}{\tau |\log(\tau)|}\right) dy.$$

With the substitution  $q = \frac{1}{\xi\sqrt{t}}[\log(y/(\tau |\log(\tau)|)) - \mu]$ , the lemma follows from the bound for the complementary Gaussian distribution function [59, Section 14.8].  $\square$

**Lemma 3.7.** *Let  $p = 1$ . The following expansion holds for option prices as  $\tau$  tends to zero:*

$$C(k, \tau) = c_5(t, 1) \exp\left(-c_1(t, 1)h_1(\tau, p) + c_2(t, 1)h_2(\tau, p)\right) \tau^{c_3(t, 1)} |\log(\tau)|^{c_4(t, 1)} \left(1 + \mathcal{O}\left(\frac{1}{|\log(\tau)|}\right)\right),$$

with the functions  $c_1, c_2, \dots, c_5, h_1$  and  $h_2$  given in Table 1.

*Proof.* Let  $\tilde{\tau} := \tau |\log(\tau)|$ . With the substitution  $y \rightarrow y/\tilde{\tau}$  and using (3.1), the option price is given by

$$\begin{aligned} C(k, \tau) &= \int_0^\infty \text{BS}(k, y, \tau) \zeta_1(y) dy = \frac{1}{\tilde{\tau}} \int_0^\infty \text{BS}\left(k, \frac{y}{\tilde{\tau}}, \tau\right) \zeta_1\left(\frac{y}{\tilde{\tau}}\right) dy \\ &= \frac{1}{\tilde{\tau}} \left\{ \int_0^L \text{BS}\left(k, \frac{y}{\tilde{\tau}}, \tau\right) \zeta_1\left(\frac{y}{\tilde{\tau}}\right) dy + \int_L^\infty \text{BS}\left(k, \frac{y}{\tilde{\tau}}, \tau\right) \zeta_1\left(\frac{y}{\tilde{\tau}}\right) dy \right\} =: \underline{C}(k, \tau) + \overline{C}(k, \tau), \end{aligned}$$

for some  $L > 0$ . Consider the first term. Using Lemma A.1 with  $\tilde{\tau} = \tau |\log(\tau)|$ , we have, as  $\tau$  tends to zero,

$$\text{BS}\left(k, \frac{y}{\tau |\log(\tau)|}, \tau\right) = \exp\left(-\frac{k^2 |\log(\tau)|}{2y} + \frac{k}{2}\right) \frac{y^{3/2}}{k^2 |\log(\tau)|^{3/2} \sqrt{2\pi}} \left[1 + \mathcal{O}\left(\frac{1}{|\log(\tau)|}\right)\right].$$

Therefore

$$\begin{aligned} \underline{C}(k, \tau) &= \frac{e^{k/2} \left(1 + \mathcal{O}\left(\frac{1}{|\log(\tau)|}\right)\right)}{|\log(\tau)|^{3/2} \xi k^2 2\pi \sqrt{t}} \int_0^L \exp\left(-\frac{k^2 |\log(\tau)|}{2y} - \frac{\left(\log\left(\frac{y}{\tau |\log(\tau)|}\right) - \mu\right)^2}{2\xi^2 t}\right) y^{1/2} dy \\ &= \exp\left(\frac{k}{2} - \frac{(\log(\tau) + \log |\log(\tau)|)^2 + \mu^2}{2\xi^2 t} - \frac{\mu(\log(\tau) + \log |\log(\tau)|)}{\xi^2 t}\right) \frac{I_1(\tau) \left[1 + \mathcal{O}\left(\frac{1}{|\log(\tau)|}\right)\right]}{\xi k^2 2\pi \sqrt{t} |\log(\tau)|^{3/2}}, \end{aligned}$$

where  $I_1(\tau) := \int_0^L g_2(y) \exp(-g_0(y) |\log \tau| + g_1(y) \log |\log(\tau)|) dy$  with  $g_0$  and  $g_1$  defined in (2.11) and

$$g_2(y) := \sqrt{y} \exp\left(\frac{\mu \log(y)}{\xi^2 t}\right).$$

The dominant contribution from the integrand is the  $|\log(\tau)|$  term; the minimum of  $g_0$  is attained at  $y^*$  given in (2.10), and  $g_0''(y^*) = 4/(\xi^6 t^3 k^4) > 0$ . Set

$$I_0(\tau) := \int_{-\infty}^\infty \exp\left(-\frac{1}{2} \left((y - y^*) \sqrt{|\log(\tau)| g_0''(y^*)} - \frac{g'(y^*) \log |\log(\tau)|}{\sqrt{|\log(\tau)| g_0''(y^*)}}\right)^2\right) dy = \sqrt{\frac{2\pi}{g_0''(y^*) |\log(\tau)|}}.$$

Then for some  $\varepsilon > 0$  as  $\tau$  tends to zero, with  $L > y^*$ ,

$$\begin{aligned}
I_1(\tau) &\sim \int_{y^*-\varepsilon}^{y^*+\varepsilon} g_2(y) \exp \left\{ -g_0(y) |\log(\tau)| + g_1(y) \log |\log(\tau)| \right\} dy \\
&\sim g_2(y^*) e^{-g_0(y^*) |\log(\tau)| + g_1(y^*) \log |\log(\tau)|} \int_{y^*-\varepsilon}^{y^*+\varepsilon} e^{-\frac{1}{2} g_0''(y^*) (y-y^*)^2 |\log(\tau)| + g_1'(y^*) (y-y^*) \log |\log(\tau)|} dy \\
&\sim g_2(y^*) \exp \left( -g_0(y^*) |\log(\tau)| + g_1(y^*) \log |\log(\tau)| + \frac{(g_1'(y^*) \log |\log(\tau)|)^2}{2g_0''(y^*) |\log(\tau)|} \right) I_0(\tau) \\
&= g_2(y^*) \exp \left( -g_0(y^*) |\log(\tau)| + g_1(y^*) \log |\log(\tau)| + \frac{(g_1'(y^*) \log |\log(\tau)|)^2}{2g_0''(y^*) |\log(\tau)|} \right) \sqrt{\frac{2\pi}{g_0''(y^*) |\log(\tau)|}}.
\end{aligned}$$

where again the  $\sim$  approximations here are exactly of the same type as in [32], and we refer the interested reader to this paper for details. Therefore as  $\tau$  tends to zero:

$$\underline{C}(k, \tau) = c_5(t, 1) \exp \left( -c_1(t, 1) h_1(\tau, 1) + c_2(t, 1) h_2(\tau, 1) \right) \tau^{c_3(t, 1) |\log(\tau)|^{c_4(t, 1)}} \left[ 1 + \mathcal{O} \left( \frac{1}{|\log(\tau)|} \right) \right],$$

with the functions  $c_1, c_2, \dots, c_5, h_1$  and  $h_2$  given in Table 1. For ease of computation we note that

$$c_5(t, 1) = \frac{\sqrt{y^*} \exp \left( \frac{k}{2} - \frac{\mu^2}{2\xi^2 t} + \frac{\mu \log(y^*)}{\xi^2 t} \right)}{k^2 \xi \sqrt{2\pi t} \sqrt{g_0''(y^*)}} = \frac{|k| \xi^3 t^{3/2} \exp \left( \frac{k}{2} - \frac{\mu^2}{2\xi^2 t} + \frac{\mu \log(y^*)}{\xi^2 t} \right)}{4\sqrt{\pi}}.$$

Now by Lemma 3.6,

$$\begin{aligned}
\overline{C}(k, \tau) &= \frac{1}{\tau |\log(\tau)|} \int_L^\infty \text{BS} \left( k, \frac{y}{\tau |\log(\tau)|}, \tau \right) \zeta_1 \left( \frac{y}{\tau |\log(\tau)|} \right) dy \\
&= \frac{1}{\tau |\log(\tau)|} \mathcal{O} \left( \exp \left\{ -\frac{1}{2\xi^2 t} \left[ \log \left( \frac{L}{\tau |\log(\tau)|} \right) - \mu \right]^2 \right\} \right).
\end{aligned}$$

Since for some  $B > 0$  we have that

$$\exp \left( -\frac{1}{2\xi^2 t} \left[ \log \left( \frac{L}{\tau |\log(\tau)|} \right) - \mu \right]^2 \right) \leq B (\tau |\log(\tau)|)^{\frac{1}{2\xi^2 t} (\log(L) - \mu)} \exp \left( -\frac{1}{2\xi^2 t} h_1(\tau, 1) \right),$$

choosing  $L$  such that  $\log(L) > \mu$  yields

$$\mathcal{O} \left( \exp \left\{ -\frac{1}{2\xi^2 t} \left[ \log \left( \frac{L}{\tau |\log(\tau)|} \right) - \mu \right]^2 \right\} \right) = \mathcal{O} \left( \exp \left( -\frac{1}{2\xi^2 t} h_1(\tau, 1) \right) \right).$$

Hence  $\overline{C}(k, \tau)$  is then exponentially subdominant to the compact part since

$$\exp \left( c_1(t, 1) h_1(\tau, 1) - c_2(t, 1) h_2(\tau, 1) \right) \mathcal{O} \left( \exp \left\{ -\frac{1}{2\xi^2 t} \left[ \log \left( \frac{L}{\tau |\log(\tau)|} \right) - \mu \right]^2 \right\} \right) = \mathcal{O} \left( e^{-c_2(t, 1) h_2(\tau, 1)} \right),$$

and the result follows.  $\square$

**3.2. Proof of Theorem 2.8.** Lemma A.2 and (3.1) yield the following asymptotics as  $\tau$  tends to infinity:

$$(3.4) \quad C(k, \tau) = 1 - m_t + m_t(1 - e^k)^+ + \tau^{-1/2} e^{k/2} \mathfrak{L}(\tau) (1 + \mathcal{O}(\tau^{-1})),$$

where  $\mathfrak{L}(\tau) := \int_0^\infty q(z) e^{-\tau z} dz$ , and we set  $q(z) \equiv -8\zeta_p(8z)/\sqrt{\pi z}$ . As  $z$  tends to zero recall the following asymptotics for the modified Bessel function of the first kind of order  $\eta$  [1, Section 9.6.10]:

$$I_\eta(z) = \frac{1}{\Gamma(\eta + 1)} \left( \frac{z}{2} \right)^\eta (1 + \mathcal{O}(z^2)).$$

Using this asymptotic and the definition of the density in (2.5) we obtain the following asymptotics for the density as  $y$  tends to zero when  $p < 1$  and absorption at the origin when  $p < 1/2$ :

$$(3.5) \quad \zeta_p(y) = \frac{y_0 y^{1-2p}}{|1-p|\xi^2 t \Gamma(|\eta|+1) (2(1-p)^2 \xi^2 t)^{|\eta|}} \exp\left(-\frac{y_0^{2(1-p)}}{2\xi^2 t (1-p)^2}\right) \left(1 + \mathcal{O}\left(y^{2(1-p)}\right)\right).$$

Analogous arguments yield that when  $p < 1/2$  and the origin is reflecting, then, as  $y$  tends to zero,

$$(3.6) \quad \zeta_p(y) = \frac{y^{-2p}}{|1-p|\xi^2 t \Gamma(\eta+1) (2(1-p)^2 \xi^2 t)^\eta} \exp\left(-\frac{y_0^{2(1-p)}}{2\xi^2 t (1-p)^2}\right) \left(1 + \mathcal{O}\left(y^{2(1-p)}\right)\right).$$

In order to apply Watson's lemma [51, Part 2, Chapter 2] to  $\mathfrak{L}$ , it is sufficient to require that the function  $q$  satisfies  $q(z) = \mathcal{O}(e^{cz})$  for some  $c > 0$  as  $z$  tends to infinity. This clearly holds here since  $\lim_{z \uparrow \infty} \zeta_p(z) = 0$ . We also require that  $q(z) = a_0 z^l (1 + \mathcal{O}(z^n))$  as  $z$  tends to zero for some  $l > -1$  and  $n > 0$ . When  $p \geq 1$ , it can be shown that  $\zeta_p$  is exponentially small, and a different method needs to be used. When  $p < 1$  and the density is as in (3.5) then  $l = 1 - 2p - 1/2$  and so we require  $p < 3/4$ . Analogously, when  $p < 1/2$  and the density is (3.6) then  $l = -2p - 1/2$  and we require  $p < 1/4$ . An application of Watson's Lemma in conjunction with (3.4) then yields Theorem 2.8.

#### APPENDIX A. BLACK-SCHOLES ASYMPTOTICS

This appendix gathers some useful expansions for the Black-Scholes Call price function BS defined in (2.8).

**Lemma A.1.** *Let  $k, y > 0$  and  $\tilde{\tau} : (0, \infty) \rightarrow (0, \infty)$  be a continuous function such that  $\lim_{\tau \downarrow 0} \frac{\tau}{\tilde{\tau}(\tau)} = 0$ . Then*

$$\text{BS}\left(k, \frac{y}{\tilde{\tau}(\tau)}, \tau\right) = \frac{y^{3/2}}{k^2 \sqrt{2\pi}} \left(\frac{\tau}{\tilde{\tau}(\tau)}\right)^{3/2} \exp\left(-\frac{k^2 \tilde{\tau}(\tau)}{2y \tau} + \frac{k}{2}\right) \left\{1 + \mathcal{O}\left(\frac{\tau}{\tilde{\tau}(\tau)}\right)\right\}, \quad \text{as } \tau \text{ tends to zero.}$$

*Proof.* Let  $k, y > 0$  and set  $\tau^*(\tau) \equiv \tau/\tilde{\tau}(\tau)$ . By assumption,  $\tau^*(\tau)$  tends to zero as  $\tau$  approaches zero, and hence (2.8) implies

$$\text{BS}\left(k, \frac{y}{\tilde{\tau}(\tau)}, \tau\right) = \text{BS}(k, y, \tau^*(\tau)) = \mathcal{N}(d_+^*(\tau)) - e^k \mathcal{N}(d_-^*(\tau)),$$

where we set  $d_\pm^*(\tau) := -k/(\sqrt{y\tau^*(\tau)}) \pm \frac{1}{2}\sqrt{y\tau^*(\tau)}$ , and  $\mathcal{N}$  is the standard normal distribution function. Note that  $d_\pm^*$  tends to  $-\infty$  as  $\tau$  tends to zero. The asymptotic expansion  $1 - \mathcal{N}(z) = (2\pi)^{-1/2} e^{-z^2/2} (z^{-1} - z^{-3} + \mathcal{O}(z^{-5}))$ , valid for large  $z$  ([1, page 932]), yields

$$\begin{aligned} \text{BS}\left(k, \frac{y}{\tilde{\tau}(\tau)}, \tau\right) &= \mathcal{N}(d_+^*(\tau)) - e^k \mathcal{N}(d_-^*(\tau)) = 1 - \mathcal{N}(-d_+^*(\tau)) - e^k (1 - \mathcal{N}(-d_-^*(\tau))) \\ &= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}d_+^*(\tau)^2/2\right) \left\{\frac{1}{d_-^*(\tau)} - \frac{1}{d_+^*(\tau)} + \frac{1}{d_+^*(\tau)^3} - \frac{1}{d_-^*(\tau)^3} + \mathcal{O}\left(\frac{1}{d_+^*(\tau)^5}\right)\right\}, \end{aligned}$$

as  $\tau$  tends to zero, where we used the identity  $\frac{1}{2}d_-^*(\tau)^2 - k = \frac{1}{2}d_+^*(\tau)^2$ . The lemma then follows from the following expansions as  $\tau$  tends to zero:

$$\begin{aligned} \exp\left(-\frac{1}{2}d_+^*(\tau)^2\right) &= \exp\left(-\frac{k^2}{2y\tau^*} + \frac{k}{2}\right) (1 + \mathcal{O}(\tau^*(\tau))), \\ \frac{1}{d_-^*(\tau)} - \frac{1}{d_+^*(\tau)} + \frac{1}{d_+^*(\tau)^3} - \frac{1}{d_-^*(\tau)^3} &= \frac{y^{3/2}\tau^*(\tau)^{3/2}}{k^2} (1 + \mathcal{O}(\tau^*(\tau))). \end{aligned}$$

□

**Lemma A.2.** *Let  $y > 0$  and  $k \in \mathbb{R}$ . Then, as  $\tau$  tends to infinity,*

$$\text{BS}(k, y, \tau) = 1 - \frac{4}{\sqrt{2\pi\tau y}} \exp\left(-\frac{y\tau}{8} + \frac{k}{2}\right) (1 + \mathcal{O}(\tau^{-1})).$$

*Proof.* Let  $y > 0$ . Then

$$\text{BS}(k, y, \tau) = \mathcal{N}(d_+^*(\tau)) - e^k \mathcal{N}(d_-^*(\tau)),$$

where we set  $d_\pm^*(\tau) := -k/(\sqrt{y\tau}) \pm \frac{1}{2}\sqrt{y\tau}$ , and  $\mathcal{N}$  is the standard normal distribution function. Hence  $d_\pm^*$  tends to  $\pm\infty$  as  $\tau$  tends to infinity. The asymptotic expansion  $1 - \mathcal{N}(z) = (2\pi)^{-1/2}e^{-z^2/2}(z^{-1} - z^{-3} + \mathcal{O}(z^{-5}))$ , valid for large  $z$  ([1, page 932]), yields

$$\begin{aligned} \text{BS}(k, y, \tau) &= \mathcal{N}(d_+^*(\tau)) - e^k (1 - \mathcal{N}(-d_-^*(\tau))) \\ &= 1 - \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}d_+^*(\tau)^2\right) \left\{ \frac{1}{d_+^*(\tau)} - \frac{1}{d_-^*(\tau)} + \frac{1}{d_-^*(\tau)^3} - \frac{1}{d_+^*(\tau)^3} + \mathcal{O}\left(\frac{1}{d_+^*(\tau)^5}\right) \right\}, \end{aligned}$$

as  $\tau$  tends to infinity, where we used the identity  $\frac{1}{2}d_-^*(\tau)^2 - k = \frac{1}{2}d_+^*(\tau)^2$ . The lemma then follows from the following expansions as  $\tau$  tends to infinity:

$$\begin{aligned} \exp\left(-\frac{1}{2}d_+^*(\tau)^2\right) &= \exp\left(-\frac{y\tau}{8} + \frac{k}{2}\right) (1 + \mathcal{O}(\tau^{-1})), \\ \frac{1}{d_+^*(\tau)} - \frac{1}{d_-^*(\tau)} + \frac{1}{d_-^*(\tau)^3} - \frac{1}{d_+^*(\tau)^3} &= \frac{4}{\sqrt{2\pi\tau y}} (1 + \mathcal{O}(\tau^{-1})). \end{aligned}$$

□

## APPENDIX B. HEDGING

We suppose that the market filtration  $(\mathcal{F}_t)_{t \geq 0}$  does not include  $\sigma(\mathcal{V})$ ; otherwise hedging reduces to the Black-Scholes case. We also suppose for simplicity that  $\mathbb{P}(\mathcal{V} = 0) = 0$ . We set  $C_t(\tau, k) := \mathbb{E}((S_\tau - e^k)^+ | \mathcal{F}_t)$  and denote the Black-Scholes price of call options at time  $t$  by  $\text{BS}_t(S_t, y, \tau) := S_t \text{BS}(k - \log S_t, y, \tau)$  with BS defined in (2.8). Using  $\sigma(\mathcal{V}) \not\subset \mathcal{F}_t$  for all  $t \geq 0$  and (3.1) we see that

$$\begin{aligned} (B.1) \quad dC_t(\tau, k) &= \int_0^\infty d\text{BS}_t(S_t, y, \tau) \zeta_p(y) dy \\ &= \int_0^\infty \left( \partial_t \text{BS}_t(S_t, y, \tau) + \frac{1}{2} S_t^2 \partial_{ss} \text{BS}_t(S_t, y, \tau) y \right) \zeta_p(y) dy dt + S_t \int_0^\infty \partial_s \text{BS}_t(S_t, y, \tau) \sqrt{y} \zeta_p(y) dy dW_t \\ &= S_t \int_0^\infty \partial_s \text{BS}_t(S_t, y, \tau) \sqrt{y} \zeta_p(y) dy dW_t, \end{aligned}$$

where the second line follows from Itô's Lemma and the last line from the Black-Scholes PDE. Construct now a hedging portfolio with a quantity  $\phi_t$  of stocks and  $\psi_t$  of bonds,  $\pi_t = \phi_t S_t + \psi_t$ , and therefore (using the self-financing condition), since the instantaneous interest rates are null,

$$(B.2) \quad d\pi_t = \phi_t dS_t = \phi_t \sqrt{\mathcal{V}} S_t dW_t.$$

It is not possible to match dynamics between (B.1) and (B.2) since  $\phi_t$  needs to be  $\mathcal{F}_t$ -adapted. Any position in the stock will leave a residual exposure for the net portfolio. One could choose a delta position that minimises the variance of the hedged portfolio, but we leave this for future research. If one adds a call option  $C.(T, k)$  in the hedging portfolio  $\pi$ , with  $T > \tau$ , then

$$\pi_t = \phi_t S_t + \theta_t C_t(T, k) + \psi_t,$$



and  $C_t(\tau, k)$  can be perfectly hedged by choosing  $\phi_t = 0$ ,  $\psi_t = C_t(\tau, k) - \theta_t C_t(T, k)$  and

$$\theta_t = \frac{\int_0^\infty \partial_s \text{BS}_t(S_t, y, \tau) \sqrt{y} \zeta_p(y) dy}{\int_0^\infty \partial_s \text{BS}_t(S_t, y, T) \sqrt{y} \zeta_p(y) dy}.$$

Interestingly one would not hold any position in the stock.

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